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Abstract

Risk capital allocation problems have been widely discussed in the academic literature. We consider a company with multiple subunits having individual portfolios. Hence, when portfolios of subunits are merged, a diversification benefit arises: the risk of the company as a whole is smaller than the sum of the risks of the individual subunits. The question is how to allocate the risk capital of the company among the subunits in a fair way. In this paper we propose to use the Lorenz set as an allocation method. We show that the Lorenz set is operational and coherent. Moreover, we propose a set of new axioms related directly to the problem of risk capital allocation and show that the Lorenz set satisfies these new axioms in contrast to other well-known coherent methods. Finally, we discuss how to deal with non-uniqueness of the Lorenz set.

Keywords: Risk capital, Cost allocation, Lorenz undominated elements of the core, Coherent risk allocation. Egalitarian allocation.

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1 Introduction

This paper considers fair allocation of risk capital. When holding risky portfolios companies and financial institutions typically withhold a level of capital, which is invested safely and acts as a buffer against unfavorable events - so-called *risk capital*. It can be used internally to measure the performance of the company itself and to ensure that the future value of the company is acceptable to the owner, CEO, chief risk officer or others. Moreover, in many cases regulation will require companies to withhold a minimum amount of risk capital in order to save the company from bankruptcy in case of e.g. financial crisis. Holding an amount of riskless capital in this case is therefore an obligation.

Often a company consists of several subunits which all contribute to the company's overall risk profile. In this paper we consider a situation where each subunit has its own unique financial portfolio with own unique risk profile. More generally, though, each subunit could have any other type of activity involving risk such as, e.g., insurance. The question is how the risk capital of the company as a whole should be allocated among the independent subunits. This question is far from trivial since the activities in the different subunits are not perfectly correlated, so usually diversification benefits arise. Allocating the risk capital solely based on each subunits' individual risk profile can be misleading, as a subunit might seem risky when looking only at the individual risk profile, but at the same time can be useful in hedging other subunits' activities. When the company's risk capital is properly allocated, each subunit's performance evaluation can be based on its ratio between expected return and allocated risk capital, i.e., the so-called Return on Risk Adjusted Capital (RORAC) approach (see e.g., [16]). The company thus gets a better overview of the overall financial situation, which helps to clarify which activities of certain subunits that create most value for the company.

Each subunit wants to minimize its share of the risk capital simply because they would rather invest in activities that give favorable returns instead of being required to withhold an amount with no or very low return. The situation is therefore similar to a classical cost allocation problem modeled as a transferable utility game, see e.g., Hougaard [19]. The seminal paper on game theoretic risk capital allocation by Denault [13] focuses on the well known Shapley and Aumann-Shapley cost allocation methods, and submits that a desirable allocation method should be, what is dubbed, *coherent*, i.e., that the resulting allocation should satisfy the stand-alone core conditions as well as a certain symmetry requirement. Since the Shapley value may fail the stand-alone core conditions it is not coherent while well-known solution concepts like the *nucleolus* (Schmeidler [24]) and the *Aumann-Shapley value*[1] (Aumann and Shapley [6]) both are examples of coherent allocation rules.

Several other papers analyze risk capital allocation from a game theoretic and axiomatic viewpoint. For instance, recently Chen et al., [10] consider the systematic risk of an entire economy and how to attribute risk to individual companies. Most other papers consider risk capital allocation between subunits at a company level. For example, Tsanakas and Barnett [28] and Boonen et. al. [9] focus on the Aumann-Shapley rule and in the latter case, generalized weighted versions of this rule. Gulick et al., [17] suggest to use a version of the nucleolus with a different notion of coalitional excess. Balog et al., [7] and Homburg and Scherpereel [18] focus on comparisons of several relevant allocation rules, including the Shapley value, the nucleolus and the Cost-Gap and β-method. In [18] they specifically look at the risk measure value-at-risk (VaR, which is not coherent) and demonstrate by an experiment that decision makers tend to disregard stand-alone core conditions and prefer simple methods like the β-method. Csóka et al., [11] consider the formal relation between the class of risk allocation games and the class of totally balanced as well as exact games.

The stand-alone core conditions (making sure that no coalition of subunits covers more than their own risk capital) are often considered the fundamental fairness requirement of any allocation method - especially when the problem itself is balanced (i.e., the core is non-empty) as in the case of risk capital allocation using a coherent risk measure (e.g., expected shortfall). We agree that relevant allocation methods should indeed be *coherent* in the sense of [13].[2] Yet, among the coherent methods we further submit that egalitarian

[1] Also known as the Euler or gradient method in the finance literature.

[2] Even if the subunits are forced to stay part of the company (at least in the short run) and hence cannot threat to block the cooperation the stand-alone core conditions are still relevant since they ensure that no coalition of subunits is subsidized by other subunits. This is particularly important if subunit managers have performance related

allocations should be preferred.

In particular, we suggest to apply the Lorenz solution (i.e., the set of Lorenz undominated allocations in the core) for risk capital allocation. Basically this solution looks for the most equally distributed allocations of risk capital subject to the stand-alone core conditions. This solution concept is well-known in game theory (see e.g., Dutta and Ray [15], Hougaard et al. [20]) but apparently it has not been analyzed in connection with risk capital allocation. We demonstrate that the Lorenz solution is coherent (with a straightforward generalization of Denault's definition of coherence to cover set valued solutions) and has further advantages over alternative well-known coherent solutions which are related directly to the problem of risk capital allocation.

We show that the Lorenz solution is the only one among the well known methods which ensures that the subunits has the right incentive to invest when such an investment is to the benefit of the company as a whole; ensures that no subunit is forced to hold risk capital when the aggregate (company) risk is zero; ensures that every subunit holds a strictly positive level of risk capital when they all have risky portfolios and a strictly positive risk capital allocation is possible given the stand-alone core conditions.

Although the Lorenz solution is not a singleton it is still operational in the sense of computation, see e.g., Smilgins [26]. Moreover selecting a single allocation from the Lorenz set is also easy since, for instance, the core allocation which minimizes the Euclidian distance to the equal split allocation will be Lorenz undominated. But of course other selection criteria tailor suited to fit the decision makers preferences could be imagined as well.

The paper is organized as follows: In Section 2 we set up the model. In Section 3 we define the Lorenz solution and records a few useful properties. In Section 4 we submit three new desirable properties directly related to risk capital allocation and demonstrates that these properties are all satisfied by the Lorenz solution but not by any of most well-known solutions from the literature on risk capital allocation (i.e., the Shalpey value, the Cost-Gap method, the nucleolus and the Aumann-Shapley value). Section 5 closes with a few final remarks.

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bonus schemes.

2 The model

In this paper we consider the allocation of risk capital along the lines of Denault [13]. Imagine a company consisting of n independent subunits. Denote by $N = \{1, \ldots, n\}$ the set of such subunits. Each subunit $i \in N$ has its own portfolio, and assume that the other subunits' portfolios are unknown to i.

At present time, say t_0, the company knows exactly how much each subunit's portfolio is worth. However, at a specific point of time in the future, say t_1, the net worth of the n portfolios are unknown. Denote by V the set of admissible portfolios and let subunit i's payoff be modeled by a random variable $X_i = r_i A_i \in V$ representing the net worth of i's investment of A_i dollars in a portfolio with a stochastic return r_i between time periods t_0 and t_1. Let $X = \{X_1, \ldots, X_n\}$ denote the companies payoff profile and let $X(S) = \sum_{i \in S} X_i$ be the payoff of the pooled portfolio of coalition $S \subseteq N$. That is, $X(N)$ is the payoff of the company as a whole at time t_1.

Risk is quantified by a *risk measure* $\rho : V \to \mathbf{R}$. In the following analysis we will always assume that the risk measure involved is a so-called *coherent* risk measure in the sense of Artzner et. al., [4]. In particular, all our examples will use *Expected Shortfall* with a degree of confidence of 5% as (coherent) risk measure and ignore all kinds of transaction costs for simplicity, see e.g., [4] or Acerbi and Tasche [1]. Expected Shortfall indicates, for each portfolio, the amount of riskless capital that should be withheld by the company in order to be able to cover expected losses given that the payoff is below a certain threshold value.

To withhold riskless capital can be considered as a cost for the company. Thus, for each coalition of subunits $S \subseteq N$, the cost associated with the payoff $X(S)$ of the pooled portfolio is defined as

$$c(S) = \rho(X(S)), \tag{1}$$

with $c(\varnothing) = 0$ per definition. We say that $c(S)$ is the *risk capital* of coalition $S \subseteq N$. In particular, $c(N)$ is total risk capital of the company that has to be allocated among the n subunits. As such, the problem can be modeled as a transferable utility (TU) game, see e.g., Peleg and Sudhölter [22].

Denote by (N, c), where N is the set of subunits and c is the cost function determined by (1), a *risk capital allocation problem*. Let Γ be the set of all such problems.

Let $Y(N,c) = \{y \in \mathbf{R}^N \mid \sum_{i\in N} y_i = c(N)\}$ be the set of possible allocations of the total risk capital $c(N)$. A *solution* on Γ is a mapping σ which associates with each problem $(N,c) \in \Gamma$ a subset $\sigma(N,c)$ of $Y(N,c)$.

One such well-known solution is *the core* given by

$$\mathcal{C}(N,c) = \{y \in Y(N,c) \mid \sum_{i\in S} y_i \leq c(S) \text{ for all } S \subseteq N\}. \tag{2}$$

The core consists of allocations of risk capital for which no coalition of subunits pay more than the risk capital associated with their pooled portfolios and thereby S does not subsidize other subunits. Mathematically speaking the core is a $n-1$ dimensional polyhedron, i.e., it is a closed and convex set with flat faces and straight edges.

By Theorem 4 in [13] it is known that if the risk measure ρ is coherent then the core of the associated risk capital problem is non-empty (i.e., the problem (N,c) is *balanced* by the Bondareva-Shapley Theorem, [22]).[3] Since (N,c) is balanced, $c(\cdot)$ is subadditive, i.e., $c(S\cup T) \leq c(S)+c(T)$ for arbitrary subsets $S,T \subseteq N$ for which $S\cap T = \varnothing$. Thus, pooling portfolios reduces the risk capital in the sense that the risk capital of the pooled portfolio is weakly smaller than the sum of the risk capital of the individual portfolios. However, Denault [13] shows that c is not *concave* (i.e., $c(S\cup T) + c(S\cap T) \leq c(S) + c(T)$ for arbitrary subsets $S,T \subseteq N$) when ρ is coherent. Hence, when using Expected Shortfall or any other coherent risk measure, the corresponding risk capital allocation game will be (totally) balanced, but not concave.

Example 1: Consider a company with four subunits $N = \{1,2,3,4\}$ whose portfolios are long 300 in SP500, long 100 in BMW stock, short 500 in oil and short 100 in Google stock respectively. Assume that we are interested in allocating the total risk capital, where the time horizon is 1 day (i.e. $t_1 - t_0 = 1$). In order to estimate the distribution of the returns, we use the historical one day returns in the period Apr 1, 2010 to Nov 1, 2012 with equal probability.[4] When ignoring any kind of transaction costs and using 5 % expected shortfall as a risk measure, the corresponding risk capital allocation problem with 2 decimal accuracy is:

$$\mathrm{c}(1) = 8.81,\ \mathrm{c}(2) = 5.08,\ \ \mathrm{c}(3) = 20.45,\ \ \mathrm{c}(4) = 3.88$$

[3] In fact, the class of risk capital allocation games given by (1) coincides with the class of totally balanced games, see Theorem 3.4 in [11].

[4] Data are downloaded from Yahoo Finance.

c(1,2) = 12.45, c(1,3) = 17.83, c(1,4) = 6.88

c(2,3) = 18.69, c(2,4) = 4.83, c(3,4) = 22.18

c(1,2,3) = 17.70, c(1,2,4) = 10.38, c(1,3,4) = 18.51, c(2,3,4) = 19.87,

c(1,2,3,4) = 17.90

The problem is to share the total risk capital of 17.90 among the four subunits. The core is illustrated in Figure 1.[5] □

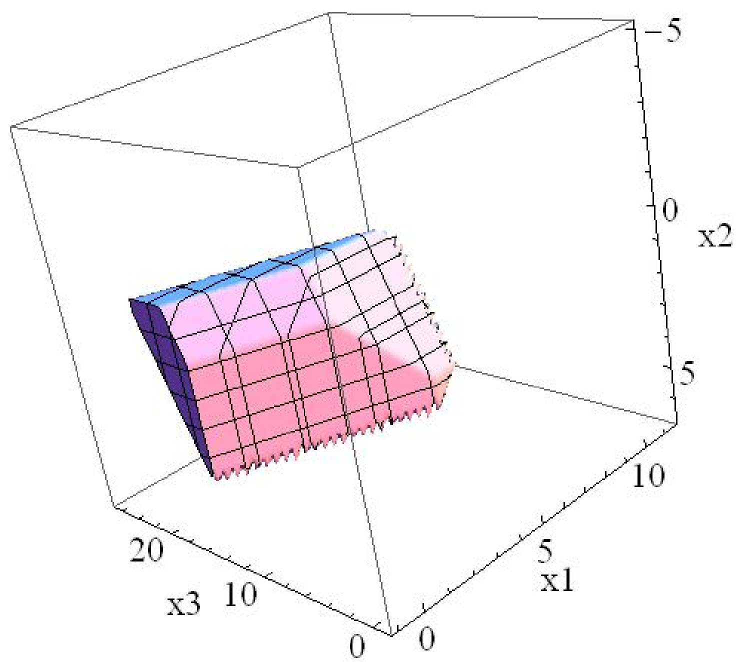

Figure 1: The core of the game. x_i is the allocated share of subunit i, $i = \{1, 2, 3\}$

[5]The core in this example is defined by 14 inequalities and one equation (the total risk capital has to sum up to exactly 17.90). Subunit 4's share can be isolated in the equation and substituted into the inequalities. Then one ends with a three dimensional problem.

3 The Lorenz solution

In the present section we shall introduce the *Lorenz set* as a solution concept for risk capital allocation. The Lorenz set is well-known in cooperative game theory (cf., e.g., Dutta and Ray [15], Dutta [13], Hougaard et al., [20], Hougaard et al., [21], Arin et al., [2], [3]), but apparently not in the context of risk capital allocation. In the next section we will argue that the Lorenz set has important advantages over other existing allocation rules discussed and analyzed in the literature on risk capital allocation.

For a given allocation $y \in \mathbf{R}^n$, define the mapping $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\varphi_k(y) = \min\{\sum_{i \in S} y_i \mid S \subseteq N, |S| = k\} \tag{3}$$

for $1 \leq k \leq n$. Now, let $y, y' \in \mathbf{R}^n$ be two allocations then y is said to Lorenz dominate y' (written $y \succ_{LD} y'$) if $\varphi(y) \geq \varphi(y')$, where $\varphi(y) \neq \varphi(y')$. With special reference to the core, an allocation y will be called *Lorenz undominated* (in the core) if there does not exist a core allocation $z \in \mathcal{C}(N,c)$ for which $z \succ_{LD} y$.

In the present paper we focus on the set of Lorenz undominated allocations in the core, i.e., the *Lorenz set*

$$L(N,c) = \{y \in \mathcal{C}(N,c) \mid \nexists\, z \in \mathcal{C}(N,c) : z \succeq_{LD} y\}. \tag{4}$$

Clearly, $L(N,c) \subseteq \mathcal{C}(N,c)$ so $L(N,c)$ is non-empty when the core is non-empty (since $\mathcal{C}(N,v)$ is compact and φ is continuous). As shown in [15], $L(N,c)$ is a singleton (i.e., $|L(N,c)| = 1$) when c is concave. Moreover, as shown in [20], $y \in L(N,c)$ if and only if there exists a strictly increasing and strictly concave function $u : \mathbf{R} \to \mathbf{R}$ such that $\sum_{i \in N} u(y_i) \geq \sum_{i \in N} u(z_i)$ for all $z \in \mathcal{C}(N,c)$

Proposition 1 below records some additional useful and well-known facts concerning the Lorenz set.

Proposition 1: *Consider a risk allocation problem* (N,c),

1. *If* $y^* = (\frac{c(N)}{n}, \ldots, \frac{c(N)}{n}) \in \mathcal{C}(N,c)$ *then* $L(N,c) = \{y^*\}$.
2. *If* $y^* = (\frac{c(N)}{n}, \ldots, \frac{c(N)}{n}) \notin \mathcal{C}(N,c)$, *then there exists a unique allocation* $z \in \mathcal{C}(N,c)$ *minimizing the Euclidian distance from* y^* *to the core, and* $z \in L(N,c)$.

3. $L(N, c)$ *is connected.*

Proof: The first statement is obvious. The second statement is noted in [3] and can easily be inferred from Theorem 2 in [20] since minimizing the Euclidian distance from a point in the core to the equal split allocation is akin to maximizing an increasing and strongly concave function subject to the core restrictions. The third statement can also be inferred from results in [20]. Q.E.D.

The facts of Proposition 1 above can be used to construct an algorithm which determines $L(N, c)$ for any given risk capital allocation problem (N, c), see Smilgins [26].

Example 1, continued: Using the algorithm of [26] we are able to determine the Lorenz set of the problem given in Example 1 above. It is given by a convex combination of the following two core allocations:

$$(5.55, 3.49, 7.52, 1.33) \text{ and } (4.46, 2.41, 8.60, 2.42).$$

This is illustrated graphically in figure 2 below. The point outside the core is the equal split and the black line on the core surface is the Lorenz set in this particular problem with the two extreme points mentioned above.

□

4 Coherent allocation rules

In this section we shall examine some desirable properties of solutions to risk allocation problems. Inspired by Denault's [13] notion of a *coherent allocation rule* (see [13]) we say that a solution σ to risk allocation problems (N, c) is *coherent* if it satisfies the following three requirements: [6]

1. $\sigma(N, c) \subseteq \mathcal{C}(N, c)$
2. If, for $i, j \in N$ with $i \neq j$, that $c(S \cup i) \leq c(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, then $y_i \leq y_j$ for $y \in \sigma(N, c)$.

[6]Note the difference between the notion of a coherent *risk measure* and a coherent *allocation rule*.

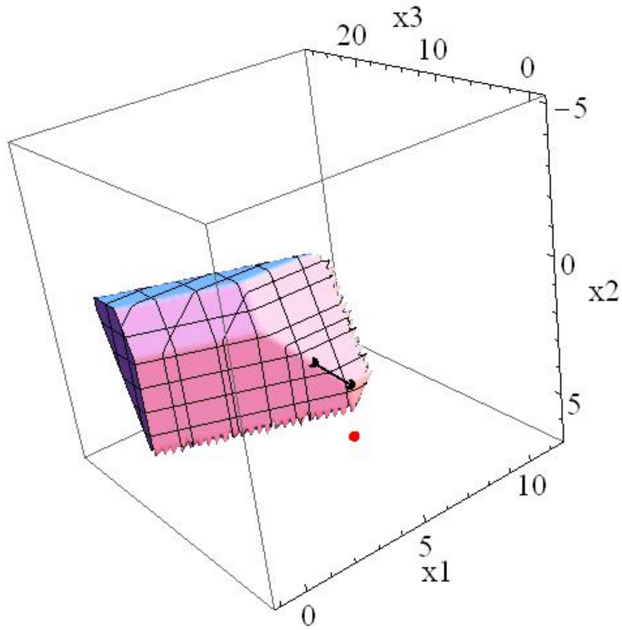

Figure 2: The Lorenz set

3. If $i \in N$ holds riskless capital k_i then $y_i = -k_i$ for $y \in \sigma(N, c)$.

In words, the first requirement states that the solution should respect the core conditions. The second requirement states that if adding sub-unit j's portfolio to any coalition of sub-units S is always more costly (in terms of risk capital) than adding sub-unit i's portfolio, then j should also pay at least as much as i when sharing the total risk capital of the company. The third requirement states that if some subunit holds a riskless portfolio it should be paid accordingly when pooled with risky portfolios.

The following observation shows that these three requirements are not independent.

Observation 2: *Any solution satisfying 1, will satisfy 3.*

Proof: Consider a subset $M \subseteq N$ of subunits i each holding riskless portfolios k_i and let $y \in \sigma(N, c)$. By 1, $\sum_{i \in M} y_i \leq \rho(\sum_{i \in M} k_i) = \sum_{i \in M} -k_i$. Moreover, since $\sum_{i \in M} y_i + \sum_{j \in N \setminus M} y_j = c(N) = c(N \setminus M) + \sum_{i \in M} -k_i$ we have that $\sum_{i \in M} y_i = c(N \setminus M) - \sum_{j \in N \setminus M} y_j + \sum_{i \in M} -k_i \geq \sum_{i \in M} -k_i$ since $c(N \setminus M) - \sum_{j \in N \setminus M} y_j \geq 0$

by 1. Thus $\sum_{i\in M} y_i = \sum_{i\in M} -k_i$. Now, by 1, $y_i \leq -k_i$ for all $i \in M$ and thus we get $y_i = -k_i$ for all $i \in M$. Q.E.D.

We can now show that the Lorenz set is basically coherent by construction.

Proposition 3: *The Lorenz set $L(N,c)$ is coherent.*

Proof: $L(N,c)$ satisfies 1, by definition. $L(N,c)$ satisfies 2, by Lemma 1 in [20]. Finally, $L(N,c)$ satisfies 3, by Observation 2 above. Q.E.D.

In the literature on risk capital allocation several other solutions (on the domain of balanced problems) are well-known and analyzed. These include; The Shapley value, the Cost-Gap rule, the nucleolus and the Aumann-Shapley rule, (for definitions, see e.g., [23], [27], [24], [6], respectively).

All these solutions are singletons, but only the latter two are coherent. Indeed, the nucleolus is a core-allocation rule (1.) and hence also satisfies 3. by Observation 2. Symmetry (2.) is also satisfied (see e.g., [22]). A serious drawback of the nucleolus is its computational complexity, especially when the number subunits becomes large.

The Aumann-Shapley rule has received considerable attention in the literature. Denault [13] shows that it is coherent when expanding the setup to allow for fractional players. The big difference between the Aumann-Shapley rule and the other (singleton) solutions in risk capital allocation problems is that while they are based on the costs/risks of all possible coalitions of players, the Aumann-Shapley rule is based on the distribution of the returns themselves. Denault [13] shows (based on Aubin [5]) that in the setup of risk capital allocation problems with Expected Shortfall as a risk measure with a degree of confidence of α, player i's allocated share is given by, $x_i^{AS} = E[-X_i \mid \sum_{i\in N} X_i \leq q_\alpha]$ where q_α is the α-quantile of the distribution of $X(N)$. One of the main drawbacks of the Aumann-Shapley rule is the fact that it is not always well defined; this happens if the quantile q_α is not a unique number (i.e., in case of non-differentiability at that point).

Example 1, continued: For the problem of Example 1, the Shapley value, the Cost-Gap rule, the nucleolus and the Aumann-Shapley rule result in the following allocations of risk capital:

$x^{Sh} = (2.43, 1.44, 13.06, 0.96)$,

$x^{CG} = (1.79, 1.67, 12.64, 1.80)$,

$x^{NUC} = (1.81, 1.14, 13.00, 1.95)$,
$x^{AS} = (-0.16, 0.04, 17.31, 0.71)$.

In all cases there is a much larger spread in payments than for $L(N, c)$ (obviously) with the Aumann-Shapley allocation as most extreme in this case; note that subunit 1 is actually paid by the other units even though all units are in fact holding risky portfolios (albeit satisfying the core conditions). □

4.1 Other desirable properties

In the following we will investigate some properties which are desirable in the particular case of risk capital allocation problems. All these properties are satisfied by the Lorenz solution, but *not* by any of the four alternative allocation rules mentioned above.

Consider a problem (N, c) and suppose that $\mathcal{C}(N, c) \cap \mathbf{R}^n_{++} \neq \emptyset$, i.e., there exists allocations in the core where all subunits pay a strictly positive amount. In particular, this implies that all coalitions of subunits hold risky portfolios, including the individual subunits themselves (i.e., $c(S) > 0$ for all $S \subseteq N$). Since each subunit is supposed to act as an independent profit maximizing investment unit, it seems reasonable to require that all individual subunits should be allocated strictly positive cost shares when covering the companies risk capital $c(N)$. This is also important in light of performance measurement based on expected return relative to allocated risk capital of the individual subunits since negative values of such performance ratios are hard to interpret, [13]. Formally,

Conditional Strict Positivity: Suppose $\mathcal{C}(N, c) \cap \mathbf{R}^n_{++} \neq \emptyset$. Then $y > 0$ for $y \in \sigma(N, c)$.

Proposition 4: *The Lorenz set $L(N, c)$ satisfies Conditional Strict Positivity.*

Proof: Let $y \in \mathcal{C}(N, c)$ and suppose there exists a set of k subunits $K \subset N$ for which $y_i < 0$ for $i \in K$. Let $\sum_{i \in K} y_i = B$. By contradiction, suppose that $y \in L(N, c)$. By Theorem 2 in [20] there exists a strictly concave function u such that y is a solution to the problem:

$$\max z(y) = u(y_1) + ... + u(y_n)$$

$$s.t. \quad y \in \mathcal{C}(N,c)$$

Let x be a another vector where:

$$x_i = y_i + |y_i| + \epsilon \text{ for all } i \in K$$

$$x_i = y_i - h_i(|B| + k\epsilon) \text{ for all } i \in N \setminus K$$

where $0 \leq h_i \leq 1$ with $\sum_{i \in N \setminus K} h_i = 1$ are chosen such that $x \in \mathcal{C}(N,c)$. Indeed, this is possible since $\mathcal{C}(N,c) \cap \mathbf{R}^n_{++} \neq \varnothing$. Because u is a strictly concave function we must have that $z(x) > z(y)$, contradicting that $y \in L(N,c)$.

Q.E.D.

The following example will demonstrate that none of the four allocation rules mentioned above satisfy Conditional Strict Positivity. Typically violations happen in case some portfolios are highly negatively correlated. In the Example 2 below, for instance, some portfolios are perfectly negatively correlated.

Example 2: Consider a company with four subunits $N = \{1,2,3,4\}$ all investing in BMW stock. The portfolios are long 295 for subunit 1 and short 100 for the other subunits. Otherwise the setup is similar to the setup in Example 1 (i.e. same risk measure, same distribution estimation, same time horizon). The corresponding risk capital allocation problem with 2 decimal accuracy is:

c(1) = 14.80, c(2) = 4.94, c(3) = 4.94, c(4) = 4.94

c(1,2) = 9.78, c(1,3) = 9.78, c(1,4) = 9.78

c(2,3) = 9.78, c(2,4) = 9.78, c(3,4) = 9.78

c(1,2,3) = 4.77, c(1,2,4) = 4.77, c(1,3,4) = 4.77, c(2,3,4) = 14.83,

c(1,2,3,4) = 0.25

First, note that $\mathcal{C}(N,c) \cap \mathbf{R}_{++} \neq \varnothing$ since, for instance, the equal split (here the unique Lorenz solution) is in the core. Yet, the four allocation rules result in the following allocations which all yield a negative cost share to subunit

1:
$x^{Sh} = (-0.02, 0.09, 0.09, 0.09)$,
$x^{CG} = (-0.38, 0.21, 0.21, 0.21)$,
$x^{NUC} = (-0.38, 0.21, 0.21, 0.21)$,
$x^{AS} = (-14.58, 4.94, 4.94, 4.94)$.

□

The next property concerns changes in allocated risk capital as a consequence of changes in invested amounts. Suppose subunit $i \in N$ invests additionally I dollars in its original portfolio such that the new payoff becomes $X_i' = r_i(A_i + I)$. The individual risk capital of subunit i therefore becomes $\rho(X_i') = \rho(X_i) + \rho(r_i I)$ because of the perfect correlation. Now, assume that this move is advantageous for the company as a whole, in the sense that the total risk capital decreases.

It is then compelling to require that this change by i cannot result in an additional share of the allocated risk capital exceeding $\rho(r_i I)$, i.e., the effect of extra investment on his own portfolio. Otherwise subunit i would not have incentive to make this additional investment to the benefit of the company as a whole. The importance of using allocation rules where the benefit of individual subunits are positively correlated with the worth of the decision to the company has been emphasized at least since Shubik [25]. Formally,

Advantageous Changes: Let (N, c) and (N, c') be two problems where c is based on payoff profile $X = \{X_1, \ldots, X_n\}$ and c' is based on payoff profile $X' = \{X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n\}$. If $c'(N) < c(N)$ then $\sigma_i(N, c') \leq \sigma_i(N, c) + \rho(r_i I)$.

Proposition 5: *The Lorenz set $L(N, c)$ satisfies Advantageous Changes.*

Proof: Consider two problems (N, c) and (N, c') as above where $c(N) > c'(N)$. By definition we have that $c'(S) = c(S)$ for all $S \not\ni \{i\}$. Moreover, $c'(\{i\}) = c(\{i\}) + \rho(r_i I)$ and for all other coalitions $\max[0, c(S) - \rho(r_i I)] \leq c'(S) \leq c(S) + \rho(r_i I)$, i.e., the risk of any coalition where i is present can at most fall by $\rho(r_i I)$. However, the risk of any coalition can at most be eliminated, which will be the case if the risk of i's portfolio and coalition $S \setminus \{i\}$'s portfolio are perfectly negatively correlated. At the same time the risk of any coalition can maximally yield an extra risk of $\rho(r_i I)$ which will be the case when the portfolios are perfectly positively correlated.

Let $d = c(N) - c'(N)$ denote the decrease in total risk capital due to i's portfolio change. Clearly, $d \leq \rho(r_i I)$. Let $x \in L(N,c)$ and $x' \in L(N,c')$ and suppose, by contradiction, that $x'_i = x_i + \rho(r_i I) + b$ where $b > 0$, i.e. i's cost share increases by more than $\rho(r_i I)$. Now, we only need to focus on situations where the core conditions allow i to get a cost share of $x'_i = x_i + \rho(r_i I) + b$. Consequently, going from the problem (N,c) to the problem (N,c') some subunits will share a total discount of $d + b + \rho(r_i I)$. Denote the set of these subunits by $T = \{j \mid x'_j < x_j\}$. Let $T_- = \{j \in T \mid x'_j < x'_i\}$ denote the set of subunits in T that gets a lower cost share than i in x'.

First, assume that $T_- \neq \varnothing$. In this case, the core conditions will allow i to transfer a fraction of b to subunits in T_-. To see this note that if we take any coalition including i, then the core conditions will not be violated since risk will be transferred within the coalition. In all the coalitions where i is not present the core conditions in (N,c) and (N,c') are the same. So, making the discount a bit smaller will not violate the core conditions either. Because i can transfer a fraction of b to subunits in T_-, we can conclude that x' is not a Lorenz allocation by Theorem 2 in [20]. Hence, the contradiction.

Second, assume that $T_- = \varnothing$. If i gets the lowest cost share among subunits in T_- in the problem (N,c'), this will also be the case in the problem (N,c), where $x_i < x'_i$ and the subunits in T do not share the total discount. But then the subunits in T_- in the problem (N,c) can transfer part of their allocated cost share to i. Thus x is not a Lorenz allocation by Theorem 2 in [20]. Indeed, the core conditions for the coalitions including i increase by no more than $\rho(r_i I)$, while i's cost share increases by $\rho(r_i I) + b$. Q.E.D.

Example 2, continued:. Assume that subunit 1 invests 7 dollars extra in its portfolio, such that it invests long 302 dollars in BMW stock. Consequently, the new problem is:

c'(1) = 15.15, c'(2) = 4.94, c'(3) = 4.94, c'(4) = 4.94

c'(1,2) = 10.14, c'(1,3) = 10.14, c'(1,4) = 10.14

c'(2,3) = 9.89, c'(2,4) = 9.89, c'(3,4) = 9.89

c'(1,2,3) = 5.12, c'(1,2,4) = 5.12, c'(1,3,4) = 5.12, c'(2,3,4) = 14.83,

c'(1,2,3,4) = 0.10

For the new problem we get:
$x^{Sh} = (0.21, -0.04, -0.04, -0.04)$,
$x^{CG} = (0.21, -0.04, -0.04, -0.04)$,
$x^{Nuc} = (0.21, -0.04, -0.04, -0.04)$,
$x^{AS} = (15.15, -5.02, -5.02, -5.02)$.

When subunit 1 invests 7 dollars extra, the Aumann-Shapley rule and the other three methods allocates subunit 1, 15.15-(-14.58)=29.73 and 0.21-(-0.38)=0.59 dollars extra, respectively. Thus, even though subunit 1's move was advantageous for the company from the perspective of risk, 1 has to pay more and even withhold an amount that is larger than the risk capital amount of the entire company itself. Subunits 2,3 and 4 are all paid by subunit 1, which will only be fair if they hold riskless portfolios. Thus, none of the four allocation rules satisfy Advantageous Changes. On the other hand, the Lorenz set does - here coinciding with the equal split, i.e.
$L(N, c') = (0.025, 0.025, 0.025, 0.025)$.

□

The next property concerns a situation where the aggregate risk capital of the company is zero. A well known fact is that when combining securities with negative correlation, the risk can be eliminated altogether. What should a fair allocation method do in this extreme case? It seems natural to suggest that no subunit should withhold any amount of risk capital, as the overall aim of the exercise is to be able to cover potential losses of the company of which there are none. Formally,

Zero Aggregate Risk: Assume all subunits in N hold risky portfolios, i.e., $c(i) > 0$ for all $i \in N$. If $c(N) = 0$ then $y = 0$ for $y \in \sigma(N, c)$.

Proposition 6: *The Lorenz set $L(N, c)$ satisfies Zero Aggregate Risk.*

Proof: If all subunits hold risky portfolios then $c(S) \geq 0$ for all $S \subseteq N$. By Proposition 1.1., $L(N, c) = \{(0, \ldots, 0)\}$. Q.E.D.

Example 2, continued:. Assume that subunit 1 now changes its invested amount to be exactly 300. In this case risk is completely eliminated. Consequently, the new problem is:

c(1) = 15.05, c(2) = 4.94, c(3) = 4.94, c(4) = 4.94

c(1,2) = 10.03, c(1,3) = 10.03, c(1,4) = 10.03

c(2,3) = 9.89, c(2,4) = 9.89, c(3,4) = 9.89

c(1,2,3) = 5.02, c(1,2,4) = 5.02, c(1,3,4) = 5.02, c(2,3,4) = 14.83,

c(1,2,3,4) = 0

The Aumann-Shapley rule here is undefined while the Shapley value, the Cost-Gap and the nucleolus all result in the same allocation:
$x^{Sh} = x^{CG} = x^{Nuc} = (0.11, -0.04, -0.04, -0.04)$.

Thus, Zero Aggregate Risk is not satisfied by any of the four methods. On the other hand, the Lorenz set consists of one point, namely where no subunit get allocated any risk, i.e., $L(N, c) = (0, 0, 0, 0)$.

□

5 Final remarks

This paper introduces a well known concept in game theory, the Lorenz solution, to risk capital allocation problems. The Lorenz set is coherent (extending Denault's definition to set valued solutions) and we further demonstrated that there are many cases where the Lorenz solution give much more reasonable results than other well-known coherent allocation rules such as the Aumann-Shapley method and the nucleolus. What we mean by reasonable is here mainly embodied in our axioms Conditional Strict Positivity, Advantageous Changes and Zero Aggregate Risk - axioms which are also violated other (non-coherent) traditional methods often discussed in finance: for instance, the Cost-Gap method violates all three axioms as demonstrated by the examples above; the β-method violates Advantageous Changes and Conditional Strict Additivity.

In particular, Advantageous Changes represents a type of property ensuring proper incentives of the subunits but in a different way than the traditional monotonicity axioms. This indicates a way to avoid the type of impossibility result presented in Csóka and Pintér [12] building on Young's property of Strong Monotonicity.[7]

[7]In [12] it is shown that there is no coherent allocation rule which satisfies strong monotonicity, in the sense that if $c(S) - c(S \setminus i) \geq c'(S) - c'(S \setminus i)$ for all $S \supseteq i$ then $\sigma_i(N, c) \geq \sigma_i(N, c')$.

Choosing a set valued solution differs from standard methods by introducing a second step selection problem for the decision maker. However, various methods from the literature on Multiple Criteria Decision Making (MCDM, see e.g., Bogetoft and Pruzan, [8]) can be used to assist further selection from the Lorenz set in accordance with the decision makers preferences. For instance, it is straightforward to select the core allocation having the smallest distance to the equal split allocation which will be Lorenz undominated, but obviously more sophisticated interactive approaches can be applied. Embedding the Lorenz solution in an interactive MCDM framework is left for future research.

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