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Trouble Comes in Threes: Core stability in Minimum Cost Connection Networks*

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Abstract

We consider a generalization of the Minimum Cost Spanning Tree (MCST) model dubbed the Minimum Cost Connection Network (MCCN) model, where network users have connection demands in the form of a pair of target nodes they want connected directly, or indirectly. Given a network which satisfies all connection demands at minimum cost, the problem consists of allocating the total cost of the efficient network among its users. As such, every MCCN problem induces a cooperative cost game where the cost of each coalition of users is given by the cost of an efficient network satisfying the demand of the users in the coalition. Unlike in the MCST model we show that the core of the induced cost game in the MCCN model can be empty (without introducing Steiner nodes). We therefore consider sufficient conditions for non-empty core. Theorem 1 shows that when the efficient network and the demand graph consist of the same components, the induced cost game has non-empty core. Theorem 2 shows that when the demand graph has at most two components the induced cost game has non-empty core.

Keywords: Minimum Cost Connection Network; Minimum Cost Spanning Tree; Cost Sharing; Fair allocation; The core; Balanced games.

JEL Classification: C70, C72, D71, D85.

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1 Introduction

We consider the Minimum Cost Connection Network (MCCN) model in which agents have individual connection demands in the form of a pair of target nodes they want connected and connections are cost to build. A minimum cost connection network is a graph that provides all agents with their desired connectivity at minimum cost. Such an efficient graph is either a tree or a forest. In the specific case where all users want connection to the same node, considered as a source, the model coincides with the classic Minimum Cost Spanning Tree (MCST) model known from combinatorial optimization (e.g., Korte and Vygen, 2018). Given a cost minimizing network, we focus on fair division of its total cost among the network users.

In the special case of the MCST-model, the seminal paper by Bird (1976) suggests to solve the issue of cost allocation by mapping the MCST-problem to a cooperative game where the cost of a given coalition of users S is given by the minimum cost of connecting the members of S to the source. Bird (1976) shows that the core of the induced cooperative cost game is non-empty. That is, we can always find a way to allocate the total cost of the efficient network such that no coalition of users can gain by implementing their own sub-network. In this sense core allocations ensure network stability and thereby sustain the efficient network configuration.

However, examples in Meggido (1978) and Tamir (1991) reveal that core stability is connected to the fact that the set of users is identified by the set of nodes in the MCST model. Adding three public nodes, i.e. Steiner nodes, to a three-agent MCST problem is enough to produce instances of cost games with empty core. So the presence of “undemanded” nodes can be a source of instability in efficient network design.

Now, generalizing users’ demands to connectivity between two arbitrary target nodes, as in the MCCN model, reveals another source of potential instability - that demands are too disconnected. By a simple three-agent example we demonstrate that we can end up in situations where the core of the induced cost game is empty without adding Steiner nodes. In the example, users have completely disconnected demands: each user demands connection between a different pair of target nodes.

Our first result (Theorem 1) shows that if users’ connection demands are disconnected, then as long as the cost minimizing network is disconnected as well, in the sense that the network and the demand graph consists of the same components, then the induced cost game has non-empty core. Thus, network stability can still be ensured in situations where it is relatively costly to connect multiple clusters of users with interlinked connectivity needs.

Our second result (Theorem 2) shows that non-empty core, and thereby network stability, can also be ensured if users’ demands are sufficiently clustered in the sense that the demand

graph contains at most two components.

Finally, Lemma 2 and its proof, shows that when the demand graph is connected every MCCN will be a spanning tree making the irreducible cost matrix (Bird, 1976) well defined, and the induced irreducible cost game is concave. This extends a similar result for the MCST model (Bergantinos and Vidal-Puga, 2007).

Related literature: Issues of fair division among users sharing a common network resource have attracted much attention over the past couple of decades: see e.g. Moulin (2013, 2019) and Hougaard (2018) for recent surveys. The standard approach to fair division has been to formulate an associated cooperative game (see e.g., Peleg and Sudholter, 2007) and use solution concepts from game theory such as the core, and the Shapley value, to guide allocation of costs and revenues.

Since the seminal papers by Claus and Kleitman (1973) and Bird (1976), the minimum cost spanning tree model and its many variations have been particularly popular topics in cost and revenue sharing in networks (see e.g. Tijs et al., 2006, Bergantinos and Vidal-Puga, 2007, Bogomolnaia and Moulin, 2010, Bogomolnaia et al., 2010, Hougaard et al. 2010, Trudeau, 2012, 2013). Implementation of minimum cost spanning trees has been studied in Bergantinos and Lorenzo (2004,2005), Bergantinos and Vidal-Puga (2010), and Hougaard and Tvede (2012).

The more general MCCN model seems originally introduced in Moulin (2009, 2014) inspired by non-cooperative cost sharing network games analyzed in the computer science literature, e.g., Anshelevich et al., (2008) and Chen et al. (2010). In particular, Moulin (2009, 2014) analyzes two types of cost sharing rules satisfying core stability and routing-proofness (a user cannot lower her cost share by reporting as multiple aliases along an alternative path connection her target nodes) when the induced MCCN games are balanced. Juarez and Kumar (2013) consider Nash implementation in a game where users choose paths connecting their target nodes. Using a particular game form, Hougaard and Tvede (2015) show that the options for implementing MCCNs are much more limited than in the MCST model. Ensuring a cost minimizing network by truthful reporting now implies compromising with individual rationality. Hougaard and Tvede (2019) introduces users' limited willingness to pay for connectivity and show that welfare maximizing networks with individually rational cost allocation are both Nash, and strong Nash implementable.

2 Model

We first recall the MCCN-model (see e.g., Moulin, 2014, or Hougaard and Tvede, 2015). Let $M = \{1, \dots, m\}$ be a set of finitely many agents and \mathcal{N} a set of finitely many locations

(nodes). The set of connections (edges) between pairs of locations is $\mathcal{N}^2 = \mathcal{N} \times \mathcal{N}$. A *cost structure* C describes costs of connecting locations and is defined by a map $c : \mathcal{N}^2 \rightarrow \mathbb{R}_+$ with: $c_{jj} = 0$ for every location j ; and, $c_{jk} > 0$ for every pair of locations (j, k) with $j \neq k$. Connections are undirected so $c_{kj} = c_{jk}$ for every pair of locations (j, k) . Connection costs are constant so the network is congestion free.

Every agent $i \in M$ has a connection *demand* $D_i = (a_i, b_i) \in \mathcal{N} \times \mathcal{N}$ with $a_i \neq b_i$, where (a_i, b_i) is a pair of locations that agent i wants to have connected directly or indirectly. A *demand structure* is a collection of demands $D = (D_i)_{i \in M}$. Note that in the classic Minimum Cost Spanning Tree (MCST) model all agents demand connection to the same location (the source). Agents can therefore be identified by the set of nodes with the source as an additional (non-involved) "agent". The MCST model is therefore a special case of the MCCN model.

A *connection problem* (M, D, C) consists of a set of agents, a demand structure, and a cost structure.

Specifically, we focus on the domain of connection problems, Γ , where *all locations are demanded*, i.e., $\cup_{i \in M} D_i = \mathcal{N}$. Thus, for any problem in Γ , the number of locations is at most $2m$. If there are n locations and no two agents have the same demand, we can have at most $\frac{n(n-1)}{2}$ agents.

A *graph* g on \mathcal{N} is a set of connections $g \subset \mathcal{N}^2$. For a cost structure C , and a graph g , let $v(C, g) \geq 0$ be the total cost of the graph g

$$v(C, g) = \sum_{jk \in g} c_{jk}.$$

For a given connection problem (M, D, C) , a *Connection Network (CN)* is a graph g meeting the connection demand of every agent $i \in M$: for every agent $i \in M$ there is a path $p = \{n_1 n_2, n_2 n_3, \dots, n_{\ell-1} n_{\ell}\}$ with $n_1 = a_i$, $n_{\ell} = b_i$ and $n_j \neq n_k$ for every pair of locations (j, k) with $j \neq k$, such that $p \subseteq g$. Denote by \mathcal{CN} the set of CNs.

A *Minimal Cost Connection Network (MCCN)* is a CN that minimizes cost: that is, g is MCCN if

$$g \in \{\arg \min_{g \in \mathcal{CN}} v(C, g)\}.$$

The set of MCCNs is non-empty and finite because the set of CNs is non-empty and finite. Clearly, every MCCN is either a tree or a forest (a graph where every component is a tree).

A connection problem (M, D, C) induces a cooperative (cost) game (M, c) where, for every coalition of agents $S \subseteq M$, $c(S) = v(C_{|S}, g^S)$: with g^S being an MCCN of the S -projected connection problem $(S, D_{|S}, C_{|S})$, i.e., the problem where only connections (and their cost) between locations demanded by agents in S are considered.

By construction, the game (M, c) is *subadditive* (i.e., for every $S, T \subseteq M$ such that $S \cap T = \emptyset$, $c(S) + c(T) \geq c(S \cup T)$).

The *core* of the game (M, c) is given by the set of allocations,

$$\text{core}(M, c) = \{x \in \mathbb{R}^M \mid \sum_{i \in M} x_i = c(M), \sum_{i \in S} x_i \leq c(S), \text{ for all } S \subset M\}. \quad (1)$$

Given the set of agents M , a collection $\mathcal{B} = \{S_1, \dots, S_k\}$ of non-empty subsets of M is called *balanced* if there exists positive numbers $\delta_1, \dots, \delta_k$ such that $\sum_{j:i \in S_j} \delta_j = 1$, for all $i \in M$. By the Bondareva-Shapley theorem, $\text{core}(M, c) \neq \emptyset$ if and only if for each balanced collection, and each system of weights δ , that

$$\sum_{S \subseteq \mathcal{B}} \delta_S c(S) \geq c(M). \quad (2)$$

Games satisfying (2) are called *balanced*.

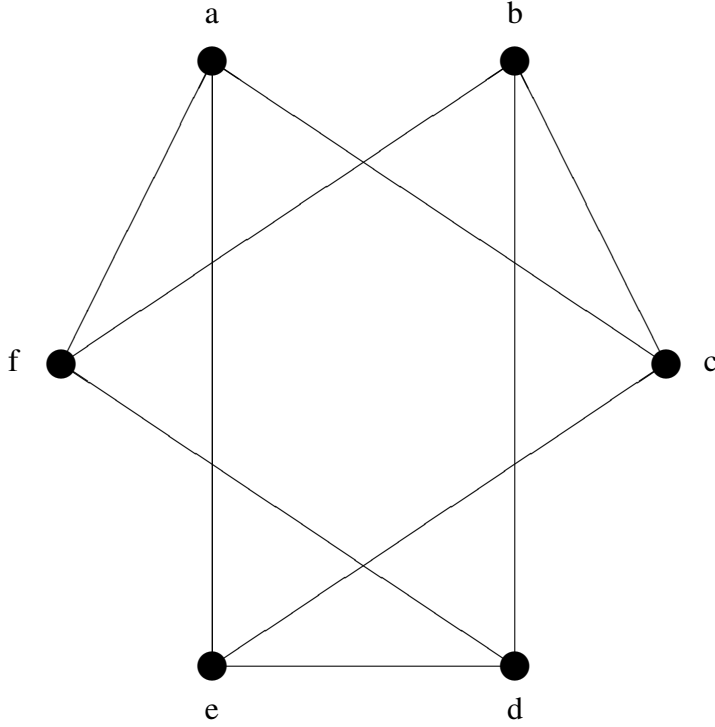
A game (M, c) is said to be *concave* if, for every $S, T \subseteq M$,

$$c(S \cup T) + c(S \cap T) \leq c(S) + c(T). \quad (3)$$

A game is concave if and only if, for each $i \in M$, i 's marginal cost $m_i(S) = c(S \cup i) - c(S)$ is non-increasing in S . Concave games are balanced.

The following example show that the core may be empty for games induced by connection problems in Γ .

Example 1: Consider six locations $\mathcal{N} = \{a, b, c, d, e, f\}$ and three agents $M = \{A, B, C\}$ with connection demands (a, b) , (c, d) , and (e, f) , respectively: so all locations are demanded. Connection costs are given as follows: $c_{af} = c_{bf} = c_{ae} = c_{de} = c_{df} = c_{ce} = c_{bc} = c_{ac} = c_{bd} = 1$, and $c_{ij} = 10$ otherwise. In the graph below only the (relevant) edges with cost equal to 1 are illustrated.



The induced cost game (M, c) has empty core since $c(AB) = c(AC) = c(BC) = 3$ and $c(ABC) = 5$ (so all agents must pay at least 2, but the total cost is only 5). \square

3 Core Stability

In this section we identify classes of connection problems for which the induced cost games are balanced (always have non-empty core). Indeed, in the special case of the MCST model, Bird (1976) demonstrated that the induced cost games are balanced.

Given a demand structure D , define the *demand graph* $G^D = \cup_{i \in M} D_i$. Problems in Γ , for which the demand graph has k components, involves at most $m + k$ nodes.

Example 1 above demonstrates that if users' demands are sufficiently disconnected, the induced cost game may have empty core. Our first result shows if the cost minimizing network is disconnected as well, in the sense that it consists of the same components as the demand graph, then core stability is still ensured.

Theorem 1 *Let $(M, C, D) \in \Gamma$. Suppose there is a MCCN g for which the number of components is equal the the number of components in G^D then (M, c) is balanced.*

The following Lemma constitutes the first step of the proof.

Lemma 1 For $(M, C, D) \in \Gamma$ suppose G^D is a tree and (M, c) is balanced. Then, for every $(M', C, D') \in \Gamma$ with $M \subset M'$ and $D_i = D'_i$ for every $i \in M$, the induced cost game is balanced.

Proof: For every $S' \subset M'$, $c(S') \geq c(S' \cap M)$ and $c(M') = c(M)$. Thus, balancedness of (M', C, D') follows from (M, C, D) being balanced. \square

With Lemma 1 we are ready to prove Theorem 1.

Proof: (of Theorem 1) Consider first $(M, D, C) \in \Gamma$ with MCCN g having as many components as G^D and where every component of g is a spanning tree. The components of G^D and g are the same since no MCCN g can “cut” a component of G^D . Denote by $\{K_1, \dots, K_l\}$ the partition of M given by the components of g and denote by g^j the cost minimizing spanning tree of the j 'th component. Thus, $|M| = m = \sum_{j=1}^l |K_j|$. We first consider the case where, for every component j , there are $|K_j|$ links in g^j , involving $|K_j| + 1$ nodes.

For every component and arbitrary coalition $S \subseteq K_j$, let κ^S be the minimum cost of satisfying the demands of agents $K_j \setminus S$ using links from the efficient graph g^j added to the demand subgraph of S , $G_S^D = \cup_{i \in S} D_i$. Note that, $\kappa^\emptyset = c(K_j)$ and $\kappa^{K_j} = 0$.

Now, for arbitrary coalitions $S \subseteq M$ consider the game defined by

$$\bar{c}(S) = c(M) - \sum_{K_j} \kappa^{S \cap K_j}. \quad (4)$$

Clearly, $c(M) = \bar{c}(M)$. We claim that $c(S) \geq \bar{c}(S)$ for every $S \subseteq M$. Indeed, suppose $c(S) < \bar{c}(S)$. Thus $c(S) < c(M) - \sum_{K_j} \kappa^{S \cap K_j} \Leftrightarrow c(S) + \sum_{K_j} \kappa^{S \cap K_j} < c(M)$, which by definition of κ contradicts that g is MCCN.

Now, we claim that the game (M, \bar{c}) is concave. Indeed, the marginal cost is given by $m_i(S) = \bar{c}(S \cup i) - \bar{c}(S) = \sum_{K_j} \kappa^{S \cap K_j} - \sum_{K_j} \kappa^{S \cup i \cap K_j}$.

Thus, if $i \in K_j$ for which $K_j \cap S = \emptyset$ then

$$m_i(S) = \max\{c_{lz} \mid lz \in g^j \text{ and } g^j - lz + a_i b_i \text{ is a spanning tree}\}$$

is constant for $T \supset S$ with $T \cap K_j = \emptyset$.

If $i \in K_j$ for which $K_j \cap S \neq \emptyset$ then $m_i(S) = \kappa^{S \cap K_j} - \kappa^{S \cup i \cap K_j}$ which by definition of κ is weakly decreasing in the size of S . Indeed, let $g^{K_j \cap S}$ denote a spanning tree obtained from g^j by replacing $|S \cap K_j|$ links in g^j with the demand graph of $S \cap K_j$ such that the total cost of the links removed is maximized. Then

$$m_i(S) = \max\{c_{lz} \mid lz \in g^{K_j \cap S} \text{ and } g^{K_j \cap S} - lz + a_i b_i \text{ is a spanning tree}\},$$

which is weakly decreasing in the size of S .

Finally, if $i \in K_j$ with $K_j \cap S = \emptyset$ but $K_j \cap T \neq \emptyset$ for $T \supset S$ then

$$\begin{aligned} m_i(S) &= \max\{c_{lz} \mid lz \in g^j \text{ and } g^j - lz + a_i b_i \text{ is a spanning tree}\} \geq \\ &\max\{c_{lz} \mid lz \in g^{K_j \cap T} \text{ and } g^{K_j \cap T} - lz + a_i b_i \text{ is a spanning tree}\} = m_i(T). \end{aligned}$$

To conclude, (M, \bar{c}) is concave and thus (M, c) is balanced.

By Lemma 1, this extends to problems where components of G^D span less than $|K_i| + 1$ nodes. \square

Our second result shows that if users' demands are not too disconnected, in the sense that the demand graph has at most two components, then core stability is also ensured.

Theorem 2 *Let $(M, C, D) \in \Gamma$. If G^D has at most two components, then (M, c) is balanced.*

For $M = \{1, 2\}$ the theorem follows directly from subadditivity of the induced cost game. For an arbitrary number of agents the theorem will be proved by Lemma 2 and 3 below.

Lemma 2 *For $(M, C, D) \in \Gamma$, if G^D is connected then (M, c) is balanced.*

Proof: Since G^D is connected any MCCN g will be a spanning tree. Use the construction of the irreducible matrix (Bird, 1976) to determine the irreducible cost \bar{c}_{ij} of each link $ij \in \mathcal{N}^2$: $\bar{c}_{ij} = \max_{lz \in p_{ij}^g} c_{lz}$ where p_{ij}^g is the unique path in g connecting nodes i and j . We claim that the irreducible game (M, \bar{c}) , where $\bar{c}(S)$ for each coalition $S \subseteq M$ is determined with respect to the irreducible cost matrix, is concave. Indeed, it is well-known that if G^D has the shape of a star then the irreducible game is concave (Bergantinos and Vidal-Puga, 2007). Fix a given agent $i \in M$ with demand $D_i = (a_i, b_i)$. Thus, $m_i(S) = \bar{c}(S \cup i) - \bar{c}(S)$ is non-increasing in the size of S if there exists $j, k \in S$ such that $a_i \in D_j$ and/or $b_i \in D_k$. So consider the case where neither a_i nor b_i is demanded by agents in S . In this case $m_i(S)$ will equal the cost of the two cheapest links between nodes demanded by agents in S and a_i, b_i . Clearly, the marginal cost of adding i to $T \supset S$ is therefore non-increasing. Thus, concavity follows in general.

Since (M, \bar{c}) , for which $\bar{c}(S) \leq c(S)$ for all $S \subset M$ and $\bar{c}(M) = c(M)$, is concave, it follows that (M, c) is balanced. \square

Lemma 3 For $(M, C, D) \in \Gamma$, if G^D has two components, then (M, c) is balanced.

Proof: Consider first $(M, C, D) \in \Gamma$ with MCCN g for which G^D has two components involving $m + 2$ nodes. Thus, G^D is a forest. By Theorem 1 we can restrict attention to the case where g is a spanning tree connecting all $m + 2$ nodes.

Consider an arbitrary MCCN g . For any coalition $S \subseteq M$, let κ^S be the minimum cost of satisfying the demands of agents $M \setminus S$ using links from g added to the demand subgraph of S , $G_S^D = \cup_{i \in S} D_i$. In particular, $\kappa^\emptyset = c(M)$ and $\kappa^M = 0$.

Now, define the game (M, \bar{c}) by

$$\bar{c}(S) = c(M) - \kappa^S \quad (5)$$

for every $S \subseteq M$. Clearly, $c(M) = \bar{c}(M)$ and $c(S) \geq \bar{c}(S)$ for all $S \subseteq M$ (suppose $c(S) < \bar{c}(S) = c(M) - \kappa^S$ then $c(S) + \kappa^S < c(M)$ which by definition of κ^S contradicts that g is MCCN).

We claim that the game (M, \bar{c}) is concave. Indeed, $m_i(S) = \bar{c}(S \cup i) - \bar{c}(S) = \kappa^S - \kappa^{S \cup i}$. Since there are two components and the MCCN g is a spanning tree

$$m_i(\emptyset) = \max\{c_{l_z} \mid l_z \in g \text{ and } g - l_z + a_i b_i \text{ is a spanning tree}\}$$

- that is, by definition of κ , agent i 's demanded connection $a_i b_i$ can replace exactly one link, $l_z \in g$ while making sure all demands remain satisfied; $m_i(\emptyset)$ is equal to the most costly such link. Note that if $a_i b_i \in g$ then $m_i(\emptyset) = c_{a_i b_i}$ (the link $a_i b_i$ "replaces" itself). Now, for any coalition $S \neq \emptyset$ denote by g^S a spanning tree obtained from g by replacing $|S|$ links in g with the demand graph of S such that the total cost of the links removed is maximized. Then

$$m_i(S) = \max\{c_{l_z} \mid l_z \in g^S \text{ and } g^S - l_z + a_i b_i \text{ is a spanning tree}\}.$$

Thus, the marginal cost of agent i is non-increasing for $T \supset S$. To conclude, (M, \bar{c}) is concave and thus (M, c) is balanced.

Note that if there are three or more components, this is no longer true as adding the demanded link $a_i b_i$ of agent i may enable deletion of more than one link whilst keeping the resulting graph a spanning tree. Therefore the game may not be concave in this case, as illustrated by Example 1.

By Lemma 1, this extends to problems where components of G^D span less than $|K_i| + 1$ nodes. \square

4 Final Remarks

So far we have focused on the domain of MCCN problems for which all locations are demanded. We close with a few remarks on the presence of Steiner nodes in the MCCN model.

As mentioned in the Introduction examples in Megiddo (1978) and Tamir (1991) imply that games with empty core can occur in problems with at least three agents where the demand graph is connected and includes the shape of a star by adding three (or more) Steiner nodes.

Yet, we conjecture that if the demand graph has the shape of a connected chain (i.e., agents' demands form a path with $b_i = a_{i+1}$ for $i = 1, \dots, m - 1$) the induced cost games are balanced even when allowing for the presence of Steiner nodes. Moreover, we conjecture that if the demand graph is connected, the induced game is balanced if at most two Steiner nodes are introduced.

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