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# Implementation of Optimal Connection Networks<sup>1</sup>

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## Abstract

We consider a connection networks model. Every agent has a demand in the form of pairs of locations she wants connected, and a willingness to pay for connectivity. A planner aims at implementing a welfare maximizing network and allocating the resulting cost, but information is asymmetric: agents are fully informed, the planner is ignorant. The options for full implementation in Nash and strong Nash equilibria are studied. We simplify strategy sets without changing the set of Nash implementable correspondences. We show the correspondence of consisting of welfare maximizing networks and individually rational cost allocations is implementable. We construct a minimal Nash implementable desirable solution in the set of upper hemi-continuous and Nash implementable solutions. It is not possible to implement solutions such as the Shapley value unless we settle for partial implementation.

**Keywords:** Connection networks; Welfare maximization; Nash Implementation; Strong Nash Implementation.

**JEL Classification:** C70, C72, D71, D85.

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# 1 Introduction

*Overview of the paper:* We consider a network design problem where connections are costly and every agent has a willingness to pay for connectivity. The problem is to implement and allocate costs of a welfare maximizing network.

By focusing on welfare maximization we are able to address the question of optimal network size because there can be agents for which the cost of including them in the network exceeds the benefit of the welfare gain. By indicating a high willingness to pay, agents can ensure that their demands are satisfied, but their resulting payments may be high. On the other hand, by indicating a low willingness to pay, agents can ensure a low payment, but their demand may not be satisfied. We analyze ways of resolving this tension.

Standard network cost sharing models, including the classic minimum cost spanning tree model (e.g., Anshelevich et al. 2008, Moulin, 2014, Trudeau, 2013) assume that every user's demand has to be satisfied. Consequently, the optimal network size becomes trivial, and the tension mentioned above is not present. By including finite willingness to pay, we bridge the network cost sharing models (originating from the operations research literature) and the network formation models from the economics literature where agents derive value from different network configurations (e.g., Jackson and Wolinsky, 1996; Mutuswami and Winter, 2002).

If agents are left to themselves to establish and allocate the cost of the network, the outcome is typically not a welfare maximizing connection network. Indeed, the core of the induced cooperative game can be empty (Meggido, 1978a; Tamir, 1991; Hougard and Tvede, 2019). Consequently, decentralized organization of networks can result in no or inefficient networks.

We therefore adopt a mechanism design approach where agents have complete information and study classic Nash implementation (see e.g., the survey in Maskin and Sjöström, 2002): a benevolent social planner wants all equilibrium outcomes to be *desirable*, in that the chosen network is *welfare maximizing* and payments are *individually rational*, for every possible state of the world (connection costs, demands, and willingness to pay for every agent). Unlike agents, the planner is ignorant about the true state of the world.

Specifically, we examine the possibility of full implementation in Nash and strong Nash equilibria. All Nash implementable solutions can be implemented by the canonical mechanism described in the proof of Theorem 2 in Maskin and Sjöström (2002). Translated directly into the present context the canonical mechanism is quite demanding in terms of the amount of information that agents have to report to the planner. We therefore start out by showing that the canonical mechanism can be modified resulting in a considerable simplification of agents' strategy sets (Theorem 1).

We then focus on the possibility of Nash implementation, and first show that it is impossible to implement (budget-balanced) cost sharing rules, such as the celebrated Shapley value, for which there is a unique distribution of the welfare generated by the network (Theorem 2). Therefore, we focus on the correspondence from states to all desirable outcomes and show that it is Nash and strong Nash implementable (Theorems 3 and 5). Since the correspondence of desirable outcomes is rather large it is natural to examine whether minimally implementable correspondences exist. A simple example demonstrates that they do not (Observation 1). Adding continuity would be appealing in terms of robustness. But welfare maximizing networks vary discontinuously with willingnesses to pay, and payments vary discontinuously with networks. Therefore we consider upper hemi-continuity instead of continuity and construct a minimal correspondence in the set of upper hemi-continuous and Nash implementable correspondences from states to desirable outcomes (Theorem 4).

Summing up, the planner on the one hand can implement welfare maximizing networks with individually rational payments, and, on the other hand has to be flexible in assigning cost shares and not use a specific cost sharing rule such as the Shapley value. Welfare gains may not be equally distributed: specifically, there is no way to ensure that all agents get a positive share. Consequently, centralized organization of networks can result in welfare maximizing networks.

Our results have a parallel interpretation concerning provision of multiple public goods (Mutuswami and Winter, 2004; Hougaard and Moulin, 2014). Every connection can be seen as a public good, and connection demands map into minimal sets of public goods providing connectivity. Agents would then report sets of such minimal service sets for every agent together with a profile of willingnesses to pay. Outcomes are sets of public goods maximizing welfare (the total willingness to pay minus total cost of provision) as well as a vector of payments respecting agents' willingness to pay.

*An illustration:* Mesopotamia in the last half of the fourth millennium BC illustrates the importance of how networks are organized. The area around Euphrates and Tigris from southern Iraq to south-eastern Turkey, including north-eastern Syria and south-western Iran, seems to have been integrated by the end of the sixth millennium BC. Artefacts indicate the culture spread from the alluvial plain in southern Iraq and mixed with local cultures. See the articles in Carter and Phillip (2006) for more on the Ubaid.

Up to around the middle of the fourth millennium BC cities such as Tell Brak in north-eastern Syria were at least as developed as cities such as Uruk in southern Iraq. But, in the last half of the fourth millennium BC cities in southern Iraq took off. The takeoff was driven by trade with raw materials flowing downstream and manufactured goods flowing upstream along Euphrates and Tigris. Cities in southern Iraq and their hinterlands were city-states of

which Uruk was dominant. Indeed, during the last half of the fourth millennium BC the ratio between the sizes of Uruk and the second largest city in southern Iraq grew from around two to more than four and a half. The city-states organized a network of settlements in the rest of the area around Euphrates and Tigris. Many of these settlements were strategically located at junctions of north-south and east-west trade routes. See Algaze (1993, 2008) and the articles in Rothman (2001) and Postgate (2002) for more on the Uruk.

At the end of the fourth millennium BC, the city-states declined and the network collapsed. Water was the sole resource in southern Iraq. It was used to make the soil quite fertile through a patchwork of irrigation channels. In Algaze (1993) it is hypothesized that salinization of the soil and strengthening of societies in other parts of the network weakened the economic viability of the city-states in southern Iraq. Consequently, the network went from being centrally organized to being decentrally organized making us guess the collapse was caused by the change in organization.

*Related literature:* For the minimum cost spanning tree model various forms of implementation have been considered. For instance, Bergantinos and Lorenzo (2004, 2005) provide an empirical example of a decentralized network formation process where agents connect sequentially to a source. Bergantinos and Vidal-Puga (2010) consider implementation of minimum cost spanning trees via a decentralized bargaining process inspired by the bidding mechanism of Perez-Castrillo and Wettstein (2001). Hougaard and Tvede (2012) consider central implementation and suggest a specific game form where agents report connection costs to a planner. This game form fully Nash implements minimum cost spanning trees using a broad class of cost allocation rules like, for instance, the Folk-solution (Bergantinos and Vidal-Puga, 2007).

Non-cooperative behavior in the more general connection networks model was initially studied in Anshelevich et al., (2008) and Chen et al., (2010). Both papers focus on equilibrium performance measured by the indices "Price of Anarchy" and "Price of Stability", i.e., the ratio between lowest (highest) welfare at equilibrium and maximal welfare. In a context where the planner is fully informed, but unable to enforce a centralized network solution, Juarez and Kumar (2013) use a game form inspired by the model in Anshelevich et al. (2008). Loosely speaking, they show that a cost allocation rule implements an efficient network (in the sense that, an efficient network is a Nash equilibrium outcome, and it Pareto dominates the outcome of all other equilibria) if and only if the allocation rule is a function of total network cost only: adding equal treatment of equals, in effect leaves the equal split rule as the only possibility.

Generalizing the game form in Hougaard and Tvede (2012) to connection networks, Hougaard and Tvede (2015) obtain similar results in a centralized setting. Full Nash imple-

mentation of an efficient network is only possible provided the planner knows the connection demand of every agent, and only under very strong assumptions on the cost allocation rule, in effect violating individual rationality. In case the planner does *not* know connection demands, truthful reporting is a Nash equilibrium implementing a cost minimizing connection network, but other equilibria may induce very inefficient networks (in fact, the "Price of Anarchy" is unbounded even if the planner has full knowledge of connection costs). It is therefore somewhat remarkable that full Nash implementation of desirable outcomes is possible in the more general version of the model where agents have limited willingness to pay for connectivity: albeit not when using a specific cost sharing rule.

Considering a network formation setting a la Jackson and Wolinsky (1996), Mutuswami and Winter (2002) show that a specific solution, namely the Shapley value, can be implemented in subgame perfect Nash equilibrium. A difference to our setting is that the planner knows connection costs. If agents' willingnesses to pay are private information it is possible to apply a so-called Moulin-mechanism (basically, a sequential auction induced by a given cost sharing rule such as the Shapley value: Moulin, 1999, Moulin and Shenker, 2001). Moulin-mechanisms are known to be group-strategyproof and budget-balanced. However, the outcome may be highly inefficient even for simple linear networks. Young (1998) presents a simple auction mechanism to implement a welfare maximizing network in strong Nash equilibrium accepting that the mechanism can produce a surplus to the planner.

## 2 The Model

In the present section we introduce our framework and discuss the model.

### Set Up

Let  $\mathcal{M} = \{1, \dots, m\}$  be a set of finitely many agents with  $m \geq 3$  and  $\mathcal{N}$  a set of finitely many locations. The set of connections between pairs of locations is  $\mathcal{N}^2 = \mathcal{N} \times \mathcal{N}$ . A *cost structure*  $C$  describes costs of connecting locations and is defined by a map  $c : \mathcal{N}^2 \rightarrow \mathbb{R}_+$  with:  $c_{jj} = 0$  for every location  $j$ ; and,  $c_{jk} > 0$  for every pair of locations  $(j, k)$  with  $j \neq k$ . Connections are undirected:  $c_{kj} = c_{jk}$  for every pair of locations  $(j, k)$ . Moreover, connections are congestion-free: costs are constant (independent of the number of agents using the connection). Let  $\mathcal{C}$  be the set of cost structures.

Every agent  $i \in \mathcal{M}$  has a *demand*  $D_i = (a_i, b_i, w_i) \in \mathcal{N} \times \mathcal{N} \times \mathbb{R}_{++}$  with  $a_i \neq b_i$ , where  $(a_i, b_i)$  is a pair of locations that agent  $i$  wants to have connected directly or indirectly and  $w_i > 0$  is the willingness to pay to have these locations connected. A *demand structure* is a collection of demands  $(D_i)_{i \in \mathcal{M}}$ . Let  $\mathcal{D}$  be the set of demand structures.

A graph  $g$  on  $\mathcal{N}$  is a set of connections  $g \subset \mathcal{N}^2$ . For a cost structure  $C$ , and a graph  $g$ , let  $v(C, g) \geq 0$  be the total cost of the graph  $g$ ,

$$v(C, g) = \sum_{jk \in g} c_{jk}.$$

A *path* between  $a$  and  $b$  is a graph  $h$  where  $h = \{n_1 n_2, n_2 n_3, \dots, n_{\ell-1} n_{\ell}\}$  with  $n_1 = a$ ,  $n_{\ell} = b$  and  $n_j \neq n_k$  for every pair of locations  $(j, k)$  with  $j \neq k$ . Let  $P_{ab}$  be the set of graphs that contains a path between  $a$  and  $b$ . For a demand structure and a graph,  $(D, g)$ , let  $M(D, g)$  be the set of agents  $i$  whose connection demands are satisfied in  $g$ :  $i \in M(D, g)$  if and only if  $g \in P_{a_i b_i}$ . For a demand structure and a graph,  $(D, g)$ , the *social welfare* is  $\sum_{i \in M(D, g)} w_i - v(C, g)$ .

For a cost structure and a demand structure,  $(C, D)$ , an *Optimal Connection Network* (OCN) is a graph  $g$  maximizing social welfare:  $g$  is an OCN if and only if for every graph  $h$ ,

$$\sum_{i \in M(D, g)} w_i - v(C, g) \geq \sum_{i \in M(D, h)} w_i - v(C, h)$$

The set of OCNs is non-empty and finite because the set of graphs is non-empty and finite. Clearly every OCN is either a tree or a forest (a collection of trees). Indeed, if a graph contains a cycle, then removing a connection in the cycle lowers total cost and does not change whether demands are satisfied or not. Since every OCN  $g$  is either a tree or a forest, there is a unique path between  $a_i$  and  $b_i$  in  $g$  for every agent  $i \in M(D, g)$ .

An *outcome* is a graph and cost shares  $(g, \pi^g)$  where  $\pi^g = (\pi_i^g)_{i \in \mathcal{M}}$  and  $\sum_i \pi_i^g = 1$ . Cost shares can be positive or negative corresponding to agents paying or being paid. Outcome  $(g, \pi^g)$  results in graph  $g$  and costs  $\pi_i^g v(C, g)$ . For a demand structure  $D$  and an outcome  $(g, \pi^g)$ , the utility of agent  $i$  is  $w_i - \pi_i^g v(C, g)$  provided her connection demand is satisfied  $g \in P_{a_i b_i}$ , and,  $-\pi_i^g v(C, g)$  provided her connection demand is not satisfied  $g \notin P_{a_i b_i}$ .

Let  $\mathcal{O}$  be the set of outcomes. A *desirable outcome* is an outcome  $(g, \pi^g)$  for which:  $g$  is an OCN; and, nobody pays more her willingness to pay  $\pi_i^g v(C, g) \leq w_i$  for every  $i \in M(D, g)$  and  $\pi_i^g \leq 0$  for every  $i \notin M(D, g)$ . A *no-subsidy (NS) desirable outcome* is a desirable outcome  $(g, \pi^g)$  where no agent is subsidized:  $\pi_i^g \geq 0$  for every  $i$  so  $\pi_i^g = 0$  for every  $i \notin M(D, g)$ . For cost structures and demand structures  $(C, D)$ , let  $\mathcal{O}^d(C, D) \subset \mathcal{O}$  be the set of desirable outcomes, and let  $\mathcal{O}_0^d(C, D) \subset \mathcal{O}^d(C, D)$  the set of NS-desirable outcomes.

A *solution*  $\Gamma : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{O}$  is a correspondence from cost structures and demand structures to outcomes. We consider two solutions: the desirable solution  $\Gamma^d$  mapping cost structures and demand structures to sets of all desirable outcomes  $\Gamma^d(C, D) = \mathcal{O}^d(C, D)$ ; and, the NS desirable solution  $\Gamma_0^d$  mapping cost structures and demand structures to sets of all NS desirable outcomes  $\Gamma_0^d(C, D) = \mathcal{O}_0^d(C, D)$ .

Desirable outcomes are appealing because they are efficient and individually rational. The networks are OCNs, their costs are exactly covered and nobody pays more than her

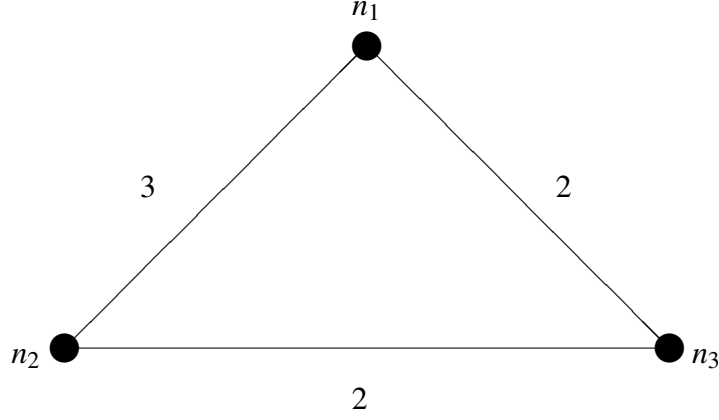


willingness to pay. In terms of fairness, individual rationality can be seen as a minimum requirement, at least nobody gets punished by realizing a desirable outcome. Adding NS to desirable outcomes can be interpreted as adding some form of independence to outcomes. Agents, whose connection demands are not satisfied, are not involved in that they neither pay nor receive anything. However, independence can be seen as going against fairness. Welfare of agents, whose connection demands are satisfied, can very well increase, but welfare of agents, whose connection demands are not satisfied, do not change. Indeed, negative payments to agents, whose connection demand is not satisfied, can ensure everybody benefits from desirable outcomes.

## Comments

Our framework enriches the standard cost allocation model in connection networks (see e.g., Bergantinos and Vidal-Puga, 2007; Anshelevich et al., 2008; Bogomolnaia, Holzman and Moulin, 2010; Bogomolnaia and Moulin, 2010; Trudeau, 2012; Moulin, 2014; Hougaard and Tvede, 2015) by adding willingness to pay to the characteristics of every agent. At first glance it might seem like a minor variation, but it introduces the fundamental question of which connection demands to satisfy. In contrast to the standard framework where all connection demands have to be satisfied, we can now compare the cost of satisfying an agent's demand with her willingness to pay. Thus, we are able to address the optimal size of the network by aiming at social welfare maximization, in contrast to the cost minimization of the standard models. Consider Example 1 below.

**Example 1:** Three locations  $\mathcal{N} = \{n_1, n_2, n_3\}$  with connection costs  $c_{12} = 3, c_{13} = c_{23} = 2$ , are shown below.



Say, two agents Ann and Bob have connection demands  $(n_1, n_2)$  and  $(n_1, n_3)$  respectively. Given that both connection demands have to be satisfied the cost minimizing network is clearly  $g = \{n_1 n_3, n_2 n_3\}$  with a total cost of 4. Now assume the agents have limited willingness to pay;  $w_A$  and  $w_B$  respectively. This changes the problem radically because now it may no longer be optimal to satisfy both Ann and Bob: if  $w_A \geq 3$  but  $w_B \leq 1$  it is welfare maximizing to exclude Bob and implement  $g' = \{n_1 n_2\}$ ; if  $w_B \geq 2$  but  $w_A \leq 2$  it is welfare maximizing to exclude Ann and implement  $g'' = \{n_1 n_3\}$ ; and if  $w_A \leq 3$ ,  $w_B \leq 2$  and  $w_A + w_B \leq 4$  it is actually welfare maximizing to exclude both agents (implementing the empty network).  $\square$

The model induces a cooperative game  $(\mathcal{M}, w)$ , where the value  $w(S)$  of every coalition  $S \subseteq \mathcal{M}$ , is naturally defined as the maximum total welfare obtainable by connecting agents in  $S$ . As demonstrated by the elegant example in Tamir (1991), the game  $(\mathcal{M}, w)$  can have empty core in case some locations are undemanded. As shown in Hougaard and Tvede (2019) the core may be empty even when all locations are demanded. Consequently, decentralized mechanisms cannot be expected to work.

### 3 Implementation

The planner aims at implementing desirable or NS-desirable outcomes and designs a game that agents play. The equilibria of the game have to be desirable or NS-desirable outcomes that the planner aims to implement. We assume the planner is ignorant, but agents know the cost structure and demand structure  $(C, D)$ . By restricting outcomes to be desirable we restrict outcomes to be welfare maximizing and individually rational in that no agent pays more than her willingness to pay.

A mechanism  $F = ((S_i)_i, f)$  consists of a strategy set for every agent,  $S_i$ , and a map from lists of individual strategies and costs to outcomes,  $f : \times_i S_i \times \mathcal{C} \rightarrow \mathcal{O}$ . A list of individual strategies  $(\bar{s}_i)_i$  is a *Nash equilibrium* provided there is no agent  $j$  and strategy  $s_j$ , such

that  $f(s_j, (\bar{s}_i)_{i \neq j})$  is strictly preferred to  $f((\bar{s}_i)_i)$  by agent  $j$ . A list of individual strategies  $(\bar{s}_i)_i$  is a *strong Nash equilibrium* provided there is no group of agents  $T \subset \mathcal{M}$ , and list of individual strategies for agents in  $T$ ,  $(s_j)_{j \in T}$ , such that  $f((s_j)_{j \in T}, (\bar{s}_k)_{k \in T^c})$  is strictly preferred to  $f((\bar{s}_i)_i)$  by every agent in  $T$ . A solution  $\Gamma : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{O}$  is *implementable* for (strong) Nash equilibrium provided there exists a mechanism  $F$  such that for all cost structures and demand structures,  $(C, D)$ , the set of (strong) Nash equilibria for  $F$  is  $\Gamma(C, D)$ .

## A mechanism for Nash implementation

In the present setting, solutions are Nash implementable if and only if they are monotonic according to Theorems 1 and 2 in Maskin and Sjöström (2002). Fundamentals are outcomes and states, where states determine preferences over outcomes. In the canonical mechanism for Nash implementation in Maskin and Sjöström (2002) every agent submits a state, an outcome and a natural number. In the present setting, every agent would submit cost structure, a demand structure, an outcome and natural number.

The strategy sets of the canonical mechanism can be reduced as shown in Saijo (1988). Fundamentals are outcomes and preferences instead of states and every agent submits preferences for themselves and another agent, an outcome and a natural number. Since preferences in the present setting depend on the cost structure and demand, every agent would have to submit a cost structure, demands for themselves and another agent, an outcome and a natural number.

We show that in the present setting all Nash implementable solutions can be implemented by use of mechanisms in which every agent submits a part of the cost structure, demands for themselves and another agents and an outcome. Before the result can be stated, the part of the cost structure every agent has to submit has to be specified. The set of connections between pairs of different locations is  $\mathcal{Q} = \mathcal{N}^2 \setminus \cup_j \{jj\}$ . For  $q \in \mathbb{N}$  defined by

$$\frac{n(n-1)}{m} \leq q < \frac{n(n-1)}{m} + 1,$$

where  $n(n-1)/m$  is the average number of connections between pairs of different locations per agent, let  $(\mathcal{Q}_i)_i$  be a cover of  $\mathcal{Q}$  with  $|\mathcal{Q}_i| = q$  and  $\mathcal{Q}_i \cap \mathcal{Q}_j \neq \emptyset$  for every  $i$  and  $j$ .

**Theorem 1** *All Nash implementable solutions  $\Gamma$  can be implemented by mechanisms  $((S_i)_i, F)$  with  $S_i = \mathbb{R}_{++}^q \times (\mathcal{N}^2 \times \mathbb{R}_{++})^2 \times \mathcal{O}$  for every  $i$ .*

*Proof:* Consider an outcome  $(g, \pi^g)$ . Let  $\Pi_i(C, D)$  be the set of outcomes  $(g, \pi^g)$  with  $g \in P_{a_i b_i}$  with  $(a_i, b_i, w_i) = D_i$  for some  $w_i$ . Then  $(g, \pi^g) \notin \Pi_i(C, D)$  if and only if  $g \notin P_{a_i b_i}$

with  $(a_i, b_i, w_i) = D_i$  for some  $w_i$ . For the utility function  $u_i(C, D, g, \pi^g)$  defined by

$$u_i(C, D, g, \pi^g) = \begin{cases} w_i - \pi_i^g v(C, g) & \text{for } (g, \pi^g) \in \Pi_i(C, D) \\ -\pi_i^g v(C, g) & \text{for } (g, \pi^g) \notin \Pi_i(C, D), \end{cases}$$

let  $L_i(C, D, g, \pi^g) \subset \mathcal{O}$  be the set of outcomes  $(h, \pi^h)$  for which the utility is lower than or equal to the utility of  $(g, \pi^g)$ ,

$$L_i(C, D, g, \pi^g) = \{ (h, \pi^h) \in \mathcal{O} \mid u_i(C, D, h, \pi^h) \leq u_i(C, D, g, \pi^g) \}.$$

A solution  $\Gamma$  is monotonic provided that for all outcomes  $(g, \pi^g) \in \mathcal{O}$  and all pairs of cost and demand structures  $(C, D), (C', D') \in \mathcal{C} \times \mathcal{D}$ ,  $(g, \pi^g) \in \Gamma(C, D)$  and  $L_i(C, D, g, \pi^g) \subset L_i(C', D', g, \pi^g)$  for every  $i$  imply  $(g, \pi^g) \in \Gamma(C', D')$ . Since the no veto power property in Maskin and Sjöström (2002) is satisfied for cost sharing problems, a solution is Nash implementable if and only if it is monotonic according to Theorems 1 and 2 in Maskin and Sjöström (2002).

For a Nash implementable solution  $\Gamma$  let a mechanism  $((S_i)_i, F)$  be described by  $S_i = \mathbb{R}_{++}^q \times (\mathcal{N}^2 \times \mathbb{R}_{++})^2 \times \mathcal{O}$  for every  $i$  and  $F : S \rightarrow \mathcal{O}$  defined as follows:

- In case there is  $(C, D, g, \pi^g) \in \mathcal{C} \times \mathcal{D} \times \mathcal{O}$  with  $(g, \pi^g) \in \Gamma(C, D)$  such that  $s_i = (C_i, D_i, D_{i+1}, g, \pi^g)$  for every  $i$ ,  $F(s) = (g, \pi^g)$ .
- In case there are  $j$  and  $(C, D, g, \pi^g) \in \mathcal{C} \times \mathcal{D} \times \mathcal{O}$  with  $(g, \pi^g) \in \Gamma(C, D)$  such that  $s_i = (C_i, D_i, D_{i+1}, g, \pi^g)$  for every  $i \neq j$ ,

$$F(s) = \begin{cases} \text{pr}_{\mathcal{O}} s_j & \text{for } \text{pr}_{\mathcal{O}} s_j \in L_j(C, D, g, \pi^g) \\ (g, \pi^g) & \text{for } \text{pr}_{\mathcal{O}} s_j \notin L_j(C, D, g, \pi^g) \end{cases}$$

- In all other cases, for  $\tilde{C}$  defined by  $\tilde{c}_{jk} = \max_i \{c_i^{jk}\}$  for every  $jk$  and  $\tilde{D}$  by  $\tilde{D}_i = D_{i-1}^i$  for every  $i$  let  $F(s) = (g_i, \pi^{g_i})$  for  $i$  chosen at random from the set

$$\left\{ i \mid \forall j : w_i^i + \sum_{k \in M(D_i^i, \tilde{D}_{-i, g_i}) \setminus \{i\}} \tilde{w}_k - v(\tilde{C}, g_i) \geq w_j^j + \sum_{k \in M(D_j^j, \tilde{D}_{-j, g_j}) \setminus \{j\}} \tilde{w}_k - v(\tilde{C}, g_j) \right\}$$

endowed with uniform distribution.

Let  $NE : \mathcal{C} \times \mathcal{O} \rightarrow S$  be the Nash equilibrium correspondence.

First it is shown that  $\Gamma(C, D) \subset F \circ NE(C, D)$ . Suppose  $s_i = (C_i, D_i, D_{i+1}, g, \pi^g)$  for every  $i$  and some  $(g, \pi^g) \in \Gamma(C, D)$ . Then

$$F(S_i, s_{-i}) = L_i(C, D, g, \pi^g).$$

Therefore  $s \in NE(C, D)$ .

Second it is shown that  $F \circ NE(C, D) \subset \Gamma(C, D)$ . In the first case, where there is  $(\bar{C}, \bar{D}, g, \pi^g) \in \mathcal{C} \times \mathcal{D} \times \mathcal{O}$  with  $(g, \pi^g) \in \Gamma(\bar{C}, \bar{D})$  such that  $s_i = (\bar{C}_i, \bar{D}_i, \bar{D}_{i+1}, g, \pi^g)$  for every  $i$ , a deviating agent  $j$  is able to move into the second case. Therefore

$$F(s_i, s_{-i}) = L_i(\bar{C}, \bar{D}, g, \pi^g).$$

If  $L_i(\bar{C}, \bar{D}, g, \pi^g) \not\subset L_i(C, D, g, \pi^g)$  for some  $i$ , then  $s$  is not a Nash equilibrium. If  $L_i(\bar{C}, \bar{D}, g, \pi^g) \subset L_i(C, D, g, \pi^g)$  for every  $i$ , then  $s$  is a Nash equilibrium. Since  $\Gamma$  is Nash implementable, it is monotonic, so  $(g, \pi^g) \in \Gamma(C, D)$ . In the second and the third cases, there is a deviating agent  $i$ , who is able to move into the third case, so

$$F(s_i, s_{-i}) = \mathcal{O}.$$

Hence  $s$  is not a Nash equilibrium. □

In comparison with Maskin and Sjöström (2002), every agent submits costs for her connections instead of a cost structure and demands for themselves and another agent instead of demands for everybody and no natural number. In comparison with Saijo (1988), every agent submits costs for their own connections instead of a cost structure and no natural number.

## Implementation in Nash equilibrium

The desirable and the NS-desirable solutions are appealing in that they maximize welfare and respect individual rationality. Two less appealing features of these solutions are that they are “big” and that they can be perceived as unfair. Indeed the NS-desirable solution maps problems to all pairs of OCNs and cost allocations where individual cost shares are bounded from below by zero and from above by the willingnesses to pay. Specifically, in case two agents have identical connection demands and willingnesses to pay it is possible that one agent pays her willingness to pay and gets utility zero and the other agent pays zero and gets utility equal to her willingness to pay. Solutions that for all pairs of problems and OCNs have a unique cost allocation are much “smaller” and can be more fair depending on how cost shares are determined. However these solutions are not Nash implementable. In particular, single-valued solutions based on cooperative games are not Nash implementable.

**Theorem 2** *Assume the planner knows the cost structure  $C$  and the connection demands  $(a_i, b_i)_i$ . Suppose a solution  $\Gamma : \mathbb{R}_{++}^m \rightarrow \mathcal{O}$  has the following properties:*

- *For all  $w$  and every  $g$ , there is either a unique or no  $\pi^g$  such that  $(g, \pi^g) \in \Gamma(w)$ .*

- For all  $w$ ,  $\Gamma(w) \subset \Gamma^d(w)$ .

Then  $\Gamma$  is not Nash implementable.

*Proof:* To show there is no solution with the imposed properties a simple counterexample is presented. There are three locations  $\{n_1, n_2, n_3\}$  with cost structure  $C = (c_{12}, c_{13}, c_{23})$  satisfying  $c_{13} > c_{12}, c_{23}$  and normalized such that  $c_{12} + c_{23} = 1$ . The  $m$  agents can be split into two groups  $T$  and  $T^c$ , where agents in  $T$  have connection demand  $(n_1, n_2)$  and total willingness to pay  $w_T$  and the agents in  $T^c$  have connection demand  $(n_1, n_3)$  and total willingness to pay  $w_{T^c}$ . Since the planner knows the cost structure and the connection demands,  $(C, D)$  is parameterized by willingnesses to pay  $w = (w_T, w_{T^c}) \in \mathbb{R}_{++}^2$ .

The network  $g = \{n_{12}, n_{23}\}$  is the unique OCN provided

$$\begin{cases} w_T + w_{T^c} > c_{12} + c_{23} & (g \text{ is strictly better than no network}) \\ w_{T^c} > c_{23} & (g \text{ is strictly better than } \{n_{12}\}) \\ w_T > c_{12} + c_{23} - c_{13} & (g \text{ is strictly better than } \{n_{13}\}) \end{cases}$$

The cost allocation for the two groups is  $\pi = (\pi_T, \pi_{T^c})$  with  $\pi_T + \pi_{T^c} = c_{12} + c_{23}$ . Then  $\pi_T \in [c_{12} + c_{23} - w_{T^c}, w_T]$  and  $\pi_{T^c} \in [c_{12} + c_{23} - w_T, w_{T^c}]$ .

Let  $w_T = c_{12} + \delta_T$  and  $w_{T^c} = c_{23} + \delta_{T^c}$ . Then the inequalities ensuring  $g$  is the unique OCN are satisfied if and only if  $\delta_T > c_{23} - c_{13}$ ,  $\delta_{T^c} > 0$  and  $\delta_T + \delta_{T^c} > 0$  so  $\delta_T$  can be negative. The cost allocation has to satisfy  $\pi^T \in [c_{12} - \delta_{T^c}, c_{12} + \delta_T]$  and  $\pi^{T^c} \in [c_{23} - \delta_T, c_{23} + \delta_{T^c}]$ . For  $(\delta_T, \delta_{T^c})$  and  $(\delta'_T, \delta'_{T^c})$  satisfying the inequalities and  $\delta_T < -\delta'_{T^c}$ , if  $(g, \pi) \in \Gamma(\delta_T, \delta_{T^c})$  and  $(g, \pi') \in \Gamma(\delta'_T, \delta'_{T^c})$ , then  $\pi_T < \pi'_T$  and  $\pi_{T^c} > \pi'_{T^c}$  because  $\Gamma(w) \subset \Gamma^d(w)$  for all  $w$ .

Suppose  $\Gamma$  is a Nash implementable solution. Then  $\Gamma$  is monotonic according to Theorem 1 in Maskin and Sjöström (2002). Therefore for all  $(\delta'', \varepsilon'')$  satisfying the inequalities and  $\delta''_T \geq \max\{\delta_T, \delta'_T\}$  as well as  $\delta''_{T^c} \geq \max\{\delta_{T^c}, \delta'_{T^c}\}$ ,  $(g, \pi), (g, \pi') \in \Gamma(\delta''_T, \delta''_{T^c})$  contradicting there is either a unique or no  $\pi^g$  such that  $(g, \pi^g) \in \Gamma(\delta''_T, \delta''_{T^c})$ .  $\square$

Fortunately, it turns out both desirable solutions are Nash implementable. From Theorems 1 and 2 in Maskin and Sjöström (2002) it follows that solutions are implementable if and only if they are Maskin monotonic. Hence, we simply show that both desirable solutions are Maskin monotonic. One example of a mechanism that works is a modified canonical mechanism as in Theorem 1, but in general the set of mechanisms that work is unknown.

**Theorem 3** *The desirable and the NS-desirable solutions are Nash implementable.*

*Proof:* Our setting fits the setting in Maskin and Sjöström (2002) with the cost and demand structures being the states. Therefore Theorem 2 in Maskin and Sjöström (2002) can be used to show that  $\Gamma^d$  and  $\Gamma_0^d$  are Nash implementable.

The terminology and notation in the proof of Theorem 1 is used. To show that  $\Gamma^d$  or  $\Gamma_0^d$  is monotonic suppose there are a pair of cost and demand structures  $(C, D)$  and  $(C', D')$  and an outcome  $(g, \pi^g)$  with  $(g, \pi^g) \in \Gamma^d(C, D)$  and  $(g, \pi^g) \notin \Gamma^d(C', D')$  or  $(g, \pi^g) \in \Gamma_0^d(C, D)$  and  $(g, \pi^g) \notin \Gamma_0^d(C', D')$ . Then for  $(C', D')$  either  $(g, \pi^g)$  is maximizing social welfare or  $(g, \pi^g)$  is not maximizing social welfare. If  $(g, \pi^g)$  is maximizing social welfare for  $(C', D')$ , then there is an agent  $i$  such that  $u_i(C', D', g, \pi^g) < 0$ . Therefore  $(\emptyset, (0, \dots, 0)) \in L_i(C, D, g, \pi^g)$  and  $(\emptyset, (0, \dots, 0)) \notin L_i(C', D', g, \pi^g)$ , so  $L_i(C, D, g, \pi^g) \not\subset L_i(C', D', g, \pi^g)$ . If  $(g, \pi^g)$  is not maximizing social welfare for  $(C', D')$ , then there is an outcome  $(h, \pi^h)$  such that  $\sum_i u_i(C', D', h, \pi^h) > \sum_i u_i(C', D', g, \pi^g)$ . Hence there is an outcome  $(h, \pi^h)$  such that  $u_i(C', D', h, \pi^h) > u_i(C', D', g, \pi^g)$  for every  $i$ . Since  $\sum_i u_i(C, D, h, \pi^h) \leq \sum_i u_i(C, D, g, \pi^g)$ , there is an agent  $i$  such that  $u_i(C, D, h, \pi^h) \leq u_i(C, D, g, \pi^g)$ . Hence  $(h, \pi^h) \in L_i(C, D, g, \pi^g)$  and  $(h, \pi^h) \notin L_i(C', D', g, \pi^g)$ , so  $L_i(C, D, g, \pi^g) \not\subset L_i(C', D', g, \pi^g)$ . To sum up,  $\Gamma^d$  and  $\Gamma_0^d$  are monotonic and consequently Nash implementable.  $\square$

## Minimal Nash implementable solutions

The desirable and the NS-desirable solutions are correspondences and they are “big”. Indeed they map  $(C, D)$ ’s to sets containing all pairs of OCNs and cost allocations where individual cost shares are bounded from below by zero and from above by willingnesses to pay. But, they are implementable. Solutions that map  $(C, D)$ ’s to a single cost allocation for every OCN are “small”, but not implementable as shown in Theorem 2. Therefore obvious questions are whether there are minimal implementable solutions and if so, how they look.

A Nash implementable solution  $\Gamma$  is minimal provided there is no other Nash implementable solution  $\Phi$  such that  $\Phi(C, D) \subset \Gamma(C, D)$  for all  $(C, D)$  and  $\Phi(C, D) \neq \Gamma(C, D)$  for some  $(C, D)$ . The following observation shows that there is no minimal solution in the full set of Nash implementable solutions.

**Observation 1** *Assume the planner knows the cost structure  $C$  and the connection demands  $(a_i, b_i)_i$ . There is no minimal Nash implementable solution with  $\Gamma(w) \subset \Gamma^d(w)$  for all  $w$ .*

*Proof:* To show there is no minimal Nash implementable solution with  $\Gamma(w) \subset \Gamma^d(w)$  for all  $w$  a simple counterexample is presented. There are two locations  $\{n_1, n_2\}$  with cost structure  $C = \{c_{12}\}$  satisfying  $c_{12} = 1$  and  $m$  agents with identical connection demands  $(n_1, n_2)$ . Then  $g = \emptyset$  is an OCN provided  $\sum_i w_i \leq c_{12}$  and  $g = \{n_{12}\}$  is an OCN provided

$\sum_i w_i \geq c_{12}$ . Suppose  $\Gamma : \mathbb{R}_{++}^m \rightarrow \mathcal{O}$  is a Nash implementable solution. Then  $\Gamma$  is monotonic according to Theorem 1 in Maskin and Sjöström (2002).

There is  $\tilde{w}$  with  $\tilde{w}_i < c_{12}$  for every  $i$  and  $\sum_i \tilde{w}_i > c_{12}$ . For  $(\tilde{g}, \tilde{\pi}^{\tilde{g}}) \in \Gamma(\tilde{w})$  let another correspondence  $\Phi : \mathbb{R}_{++}^m \rightarrow \mathcal{O}$  be defined by  $\Phi(w) = \Gamma(w) \setminus \{(\tilde{g}, \tilde{\pi}^{\tilde{g}})\}$  for all  $w$  so  $\Phi(w) \subset \Gamma(w)$  and  $\Phi(\tilde{w}) \neq \Gamma(\tilde{w})$ . Then  $\Phi$  is monotonic because  $\Gamma$  is monotonic, but possibly empty for some  $w$ . For all  $w$  with  $\sum_i w_i > c_{12}$  there is  $w' \in \mathbb{R}_{++}^m$  with  $\sum_i w'_i > c_{12}$  such that  $w'_i < \tilde{\pi}_i^{\tilde{g}}$  for some  $i$ . Therefore  $(\tilde{g}, \tilde{\pi}^{\tilde{g}}) \notin \Gamma(w')$  and  $\Gamma(w') \subset \Gamma(w)$  because  $\Gamma$  is monotonic so  $\Phi(w) \neq \emptyset$  for all  $w$  with  $\sum_i w_i > c_{12}$ . For all  $w$  with  $\sum_i w_i < c_{12}$ ,  $\Gamma(w) = (\emptyset, (0, \dots, 0))$  so  $(\emptyset, (0, \dots, 0)) \in \Gamma(w')$  for all  $w'$  with  $\sum_i w'_i \leq c_{12}$  because  $\Gamma$  is monotonic. Hence  $\Phi(w) \neq \emptyset$  for all  $w$  so  $\Phi$  is a solution and  $\Gamma$  is not minimal.  $\square$

The set of Nash implementable solutions contains less appealing solution such as the one constructed in the proof of Observation 1. Thus, it seems natural to require additional properties of solutions. In terms of robustness, continuity is an appealing property of solutions. However solutions mapping problems to pairs of OCNs and cost allocations are not continuous. The set of OCNs varies discontinuously with problems and cost allocations vary discontinuously with OCNs. The second best in terms of robustness is upper hemi-continuity. Trivially the desirable and the NS-desirable solutions are upper hemi-continuous. And there are minimal solutions in set of upper hemi-continuous and Nash implementable solutions with their graphs being contained in the graph of the NS-desirable solution.

To formalize the notion of sizes of solutions, let  $\Omega$  be a set of solutions. Then a solution  $\Gamma \in \Omega$  is  $\Omega$ -minimal provided  $\Phi(C, D) \subset \Gamma(C, D)$  implies  $\Phi(C, D) = \Gamma(C, D)$  for all  $(C, D)$  and all  $\Phi \in \Omega$ .

**Theorem 4** *There are minimal upper hemi-continuous solutions  $\Gamma$  with  $\Gamma(C, D) \subset \Gamma_0^d(C, D)$  for all  $(C, D)$ .*

*Proof:* First, an upper hemi-continuous and Nash implementable solution  $\Gamma$  with is constructed. Second, it is shown that  $\Gamma$  is minimal in the set of upper hemi-continuous and Nash implementable solutions.

Trivially if a correspondence is upper hemi-continuous in cost structures and willingnesses to pay, then it is upper hemi-continuous in cost structures and demand structures. Therefore connection demands  $(a_i, b_i)_i$  are assumed to be fixed and  $M(g)$  denotes  $M(D, g)$ . To ensure that our minimal solutions are indeed upper hemi-continuous, willingnesses to pay are allowed to be zero, so the set of willingnesses to pay is  $\mathbb{R}_+^m$ .

For every graph  $g$  let the set  $A^g \subset \mathcal{C} \times \mathbb{R}_+^m$  be the set of cost structures and lists of individual willingnesses to pay  $(C, w)$  for which  $g$  is an OCN. Then the set  $A^g$  is convex, closed and possibly empty. Trivially,  $(A^g)_g$  is a cover of  $\mathcal{C} \times \mathbb{R}_+^m$ . Let  $\mathcal{G} \subset 2^{\mathcal{N}^2}$  be a subset of



graphs such that  $(A^g)_{g \in \mathcal{G}}$  is a minimal cover of  $\mathcal{C} \times \mathbb{R}_+^m$ :  $(A^g)_{g \in \mathcal{G}}$  is a cover of  $\mathcal{C} \times \mathbb{R}_+^m$ ; and, for every  $h \in \mathcal{G}$ ,  $(A^g)_{g \in \mathcal{G} \setminus \{h\}}$  is not a cover of  $\mathcal{C} \times \mathbb{R}_+^m$ . For a group of agents  $T \subset \mathcal{M}$  let  $w_T$  be the willingnesses to pay for agents in the group  $w_T = (w_i)_{i \in T}$ .

First, for every  $g \in \mathcal{G}$  let  $\partial A^g \subset A^g$  be the set of minimal willingnesses to pay for agents in  $M(g)$ ,

$$\partial A^g = \left\{ (C, w) \in A^g \mid \forall (C, w') \in A^g : \begin{cases} w'_{M(g)} - w_{M(g)} \in -\mathbb{R}_+^{M(g)} \\ w'_{M(g)^c} - w_{M(g)^c} \in \mathbb{R}_+^{M(g)^c} \end{cases} \Rightarrow w = w' \right\}.$$

Second, for every graph  $g \in \mathcal{G}$  let  $\lambda^g : \partial A^g \rightarrow \mathbb{R}_+^m$  be defined by

$$\lambda_i^g(C, w) = \begin{cases} \frac{w_i}{\sum_{j \in M(g)} w_j} & \text{for } i \in M(g) \\ 0 & \text{for } i \in M(g)^c. \end{cases}$$

Then  $\lambda_i^g(C, w)v(C, g) \leq w_i$  for  $i \in M(g)$ , because  $\sum_{j \in M(g)} w_j \geq v(C, g)$ ,  $\lambda_i(C, w) = 0$  for  $i \in M(g)^c$  and  $\sum_i \lambda_i(C, w) = 1$ . Third, for every graph  $g \in \mathcal{G}$  let the correspondence  $\Gamma^g : A^g \rightarrow \mathcal{O}$  be defined by

$$\Gamma^g(C, w) = \left\{ (g, \pi^g) \mid \exists (C, w') \in \partial A^g : \begin{cases} w'_{M(g)} - w_{M(g)} \in -\mathbb{R}_+^{M(g)} \\ w'_{M(g)^c} - w_{M(g)^c} \in \mathbb{R}_+^{M(g)^c} \\ \pi^g = \lambda^g(C, w') \end{cases} \right\}.$$

Then  $\Gamma^g$  is non-empty, continuous and Nash implementable on  $A^g$  by construction. The solution  $\Gamma : \mathcal{C} \times \mathbb{R}_+^m \rightarrow \mathcal{O}$  defined by  $\Gamma(C, w) = \bigcup_{g \in \mathcal{G}} \Gamma^g(C, w)$  is upper hemi-continuous and Nash implementable on  $\mathcal{C} \times \mathbb{R}_+^m$  with  $\Gamma(C, w) \subset \Gamma_0^d(C, w)$  for all  $(C, w)$ .

Consider another upper hemi-continuous and Nash implementable solution  $\Phi$  with  $\Phi(C, w) \subset \Gamma(C, w)$  for all  $(C, w)$ . By construction of  $\Gamma$  there is  $(C', w') \in \partial A^g$  with

$$\begin{cases} w'_{M(g)} - w_{M(g)} \in -\mathbb{R}_+^{M(g)} \\ w'_{M(g)^c} - w_{M(g)^c} \in \mathbb{R}_+^{M(g)^c} \end{cases}$$

such that  $\lambda^g(C', w') = \pi^g$ . Consider a sequence of cost structures and lists of individual willingnesses to pay  $(C^n, (w_i^n)_{i \in \mathcal{M}})_{n \in \mathbb{N}}$  that converges to  $(C', w')$  with  $g$  being the unique OCN for  $(C^n, (w_i^n)_{i \in \mathcal{M}})$  and every  $n$ . If  $(g, \pi^n) \in \Gamma(C^n, (w_i^n)_{i \in \mathcal{M}})$  for every  $n$ , then  $\lim_{n \rightarrow \infty} \pi^n = \pi^g$  because  $\Gamma$  is upper hemi-continuous. Since  $\Phi(C^n, (w_i^n)_{i \in \mathcal{M}}) \subset \Gamma(C^n, (w_i^n)_{i \in \mathcal{M}})$  for every  $n$ , for all sequences  $(h^n, \pi^{hn})_{n \in \mathbb{N}}$  with  $(h^n, \pi^{hn}) \in \Phi(C^n, (w_i^n)_{i \in \mathcal{M}})$  for every  $n$ ,  $h^n = g$  for every  $n$  and  $\lim_{n \rightarrow \infty} \pi^{hn} = \pi^g$ . Therefore  $(g, \pi^g) \in \Phi(C', w')$  because  $\Phi$  is upper hemi-continuous.

Since  $\Phi$  is Nash implementable and consequently monotonic,  $(g, \pi^g) \in \Phi(C, w)$ . Hence  $\Gamma(C, w) \subset \Phi(C, w)$  so  $\Phi(C, w) = \Gamma(C, w)$  for all  $(C, w)$ .  $\square$

*Remark:* The solution constructed in the proof of Theorem 4 needs not be unique. Indeed, non-uniqueness could be caused by multiplicity of minimal covers of  $\mathcal{C} \times \mathbb{R}_{++}^m$  or continuous costs on  $\partial A^g$ .

## Implementation in strong Nash equilibrium

It could be possible for agents to coordinate their actions. Therefore implementation of solutions in strong Nash equilibrium is considered. Using a modified, and informationally more efficient, version of the mechanism in the proof of Theorem 3 in Maskin (1978) we show that the desirable solution  $\Gamma^d$  is strong Nash implementable.

**Theorem 5** *The desirable solution  $\Gamma^d$  is strong Nash implementable.*

*Proof:* For the mechanism implementing  $\Gamma^d$ , let the strategy set of every agent be the set of outcomes  $S_i = \mathcal{O}$  and the map from lists of individual strategies to outcomes  $f^d : S^m \rightarrow \mathcal{O}$  be

$$f^d(s) = \begin{cases} (g, \pi^g) & \text{for } s_1 = \dots = s_m = (g, \pi^g) \text{ and } (g, \pi^g) \in \mathcal{O}^d(C, D) \\ (\emptyset, (0, \dots, 0)) & \text{otherwise.} \end{cases}$$

Suppose every agent uses the strategy  $(g, \pi^g)$ . If  $(g, \pi^g)$  is a desirable outcome for the true state, then no agent has an incentive to change her strategy. If  $(g, \pi^g)$  is not a desirable outcome for the true state, then there is another strategy  $(h, \pi^h) \in \mathcal{O}^d(C, D)$  such that  $u_i(C, D, h, \pi^h) > u_i(C, D, g, \pi^g)$  for every  $i$ . Therefore the mechanism implements  $\Gamma^d$ .  $\square$

## Partial Implementation

Both the desirable and the NS-desirable solutions are partially Nash and strong Nash implementable by use of the modified mechanism in the proof of Theorem 5. The mechanism has strategy set  $S_i = \mathcal{O}$  for every agent and payoff function  $f^d : S^m \rightarrow \mathcal{O}$  defined by

$$f^d((s_i)_i) = \begin{cases} (g, \pi^g) & \text{for } s_1 = \dots = s_m = (g, \pi^g) \text{ and } (g, \pi^g) \in \mathcal{O}^d(C, D) \\ (\emptyset, (0, \dots, 0)) & \text{otherwise.} \end{cases}$$

Indeed, the mechanism fully strong Nash implements the desirable solution as shown in Theorem 5. It is straightforward to check that it partially implements the three other combinations of solutions and forms of implementation. Therefore, the price of stability is zero for all four combinations.

Since,  $s_i = (\emptyset, (0, \dots, 0))$ , for every  $i$ , is a Nash equilibrium, the price of anarchy is unbounded for Nash implementation. If social welfare is non-positive in case the connection demand of some agent  $i$  is satisfied, then  $s_i = (\emptyset, (0, \dots, 0))$ , for every  $i$ , is a strong Nash equilibrium for the NS-desirable solution. Indeed agent  $i$  has to increase her utility by changing her strategy. However, to increase the utility of agent  $i$ , her connection demand has to be satisfied or her cost share has to be negative. Neither of these alternatives are possible. Consequently the price of anarchy is unbounded. Obviously, the problem with the NS-desirable solution is that it may not be possible to transfer welfare between agents.

To sum up, the price of stability is zero for all four combinations and the price of anarchy is unbounded for all four combinations, but the one covered in Theorem 5.

## 4 Final Remarks

The canonical mechanism in Maskin and Sjöström (2002) can be simplified considerably as shown in Theorem 1. Further simplification of strategy sets does not seem possible as indicated by results in Lombardi and Yoshihara (2013) and Tatamitani (2001).

In our context, Young (1998) study a simple auction mechanism used to implement a welfare maximizing network in strong Nash equilibrium. In particular, each agent  $i$  submits a bid  $p_i \geq 0$  that she is willing to pay if supplied with her demanded connectivity (obviously this may not be equal to her willingness to pay); a set of bids is accepted if it maximizes revealed welfare; each agent with an accepted bid pays the bid  $p_i$ ; rejected agents pay nothing and do not obtain connectivity.

In the special case where the induced cooperative welfare game  $(\mathcal{M}, w)$  is convex (as for instance in case of rooted fixed trees, Megiddo, 1978b), every strong Nash equilibrium results in an efficient network where payments are budget-balanced (i.e., with no surplus to the planner). However, it is easy to find examples where the auction mechanism produces a surplus for the planner, even in the extreme where the planner extracts the full surplus from the agents.

So informational efficiency of Young's auction mechanism comes with the "price" of potential surplus extraction.

In the context of cost sharing, another challenging feature in the design of the canonical mechanisms (especially the one used in the proof of Theorem 5) is the required unanimity in reporting. If we try to loosen up on this, for instance, by allowing one report to deviate from the rest we risk that there is no Nash equilibrium since agents can team up against any single agent when reporting desired cost shares. Also, if reports are allowed a certain level of imprecision that will be a likely source of inefficiency. Mapping out a more precise relationship between the level of imprecision allowed and the resulting level of inefficiency

is left for future research.

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