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Drought and Groundwater Management[1]

By

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Abstract

This paper considers the problem of a water management authority faced with the threat of a drought that hits at an uncertain date. Three management policies are investigated: i) a laissez-faire (open-access) policy of automatic adjustment through a zero marginal private net benefit condition, ii) a policy of optimal dynamic management ignoring the threat of the drought and relying on automatic adjustments through a zero marginal social net benefit condition, iii) an economically optimal dynamic policy taking account of the threat of a drought. In particular, we show that the optimal pre-drought steady-state equilibrium stock size of water under policy iii) is smaller than under policy ii) and, hence, a precautionary stock size should not be built up prior to the drought.

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1. Introduction

Each year deficit rainfall and droughts takes a heavy toll in terms of destroyed crops, famine, fires and losses of human life. These events may range in severity from regional incidences lasting for a short period of time and confining themselves to crop destruction[4] to disastrous droughts.[5] The drought periods may be single event phenomena´s or recurring events and may be more or less expected events. Nowadays it is common to link the occurrence of droughts to climate change issues and there exist more or less efficient measures to deal with them. To the extent that such measures are in use, they may include institutionalized systems of scarce water distribution, various forms of water rationing as well as scarcity pricing of water.

The ability to withstand the negative consequences of droughts is related to the applied mechanisms for managing water extraction. In some countries water is typically priced to retail extractors on the basis of average production cost where total cost of a service is determined using historical accounting statistics. Such a system is not very responsive to a situation of water scarcity since there is no mechanism that induces reduced water extraction when the drought hits. Therefore, emergency measures are necessary and these measures may include rationing or more advanced systems of water reallocation.[6]

More advanced systems of water pricing include marginal cost pricing of scarce water resources with peak load pricing as an example.[7] Systems like these may be efficient in dealing with water scarcity but they do not necessarily solve the long-run management problems of water extraction during a drought. More elaborate systems of tradable water rights spanning all possible states of

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[4] Examples are the California drought in 1991 and the current forest fires in the United States.

[5] As the ones that took place in the Sahel region on the southern edge of Sahara in 1968-73 and in Vietnam and Ethiopia in 2012-2015.

[6] An example of an advanced allocation system is the 1991 California Drought Water Bank that efficiently alleviated acute problems of water shortage in California during the spring of 1991 by buying water from low valued agricultural uses and selling water to higher valued agriculture and municipal and industrial water uses (see Howitt, 1994 and 1998 and Hansen et al, 2013).

[7] See e.g. Hanke and Davis (1971) and (1973), Feldman (1975), Mann and Schlenger (1982) and Zarnikau (1994), Reynoud (2010), Garcia and Reynoud (2004) and Kneese (2013).

nature[8] have, however, the potential of dealing with problems of a drought if the rights are extended to cover futures markets (see e.g. Berck and Lipow, 1994, Brewer et al, 2007, Hanah and Stryjewski, 2012 and Zilberman and Schaengolf, 2005).

In this paper groundwater is treated as a renewable resource that occasionally may become a non-renewable resource due to a drought and the drought is basically treated as a stochastic event requiring optimal management under uncertainty. Long-run management may, therefore, include policies such as the keeping a precautionary stock of water or, as it turns out, the contrary. We consider a stylized problem of groundwater management under uncertainty, where there is a probability of a shortfall of precipitation followed by normal conditions. Thus, we consider optimal extraction of a groundwater resource under the assumption that the time at which a drought hits is unknown. One main result following from this analysis is that the optimal pre-drought stock size is lower when the threat of a drought is taken into account. This result can be explained by the fact that two counteracting effects exist when dealing with droughts. First, the threat of a drought implies that it is optimal to increase current extraction to capture higher net benefits before a temporary resource collapse and this can be labelled an extinction effect. Second, due to a desire of fast resource recovery after the drought is over current extraction is decreased so as to build up a buffer to be drawn upon later. This effect can be called a recovery effect. In this paper we formally show that the extinction effect dominates the recovery effect for groundwater. This implies that the current extraction increases and that the optimal pre-drought stock size decreases when the threat of a drought is included.

A considerable amount of economic groundwater literature has considered stochastic recharge for a single aquifer and in this literature there are two main strands. The first strand of literature uses optimal control theory on continuous time problems and is represented by, e.g., Tsur and Graham-Tomasi (1991), Tsur and Zemel (1995) and (2004), Roseta-Palma and Xepapadeas (2004) and Rubio and Casino (2001). The second strand of literature is based on a dynamic programming formulation of discrete time models and examples are Knapp and Olson (1995) and (1996) and Krishnamurthy (2016). In this paper we follow the first strand of literature but instead

[8] Debreu contingent markets provide an example.

of stochastic recharge we consider an unknown time for the occurrence of a drought. Thus, we seek to contribute to the existing economic literature on groundwater management under uncertainty.

The rest of the paper is organized as follows. In section 2 we describe the three policies we consider and the basic model assumptions while section 3 investigates an open-access scenario. Section 4 contains a dynamic management problem where the threat of a drought is ignored and section 5 analyzes a problem where the threat of a drought is included. In section 6 the results of a numerical simulation of the three policies are presented, and the main results and basic assumptions are discussed in section 7. Section 8 summarizes the main conclusions of the paper.

2. Policies and assumptions

Management policies

In this paper we consider three management policies of increasing sophistication: i) a laissez faire policy corresponding to open-access, ii) a policy of optimal dynamic management where the threat of a drought is ignored iii) an economically optimal policy taking account of the threat of a drought. Our focus in the following is to compare steady-state equilibrium stock sizes under these three policies, and, in particular, to investigate whether a buffer or precautionary stock of water should be built up prior to the drought to relieve the burden of the event once it hits.

The laissez faire policy corresponds to letting private agent´s select extraction levels without intervention by management authorities. This policy is identical to an unregulated optimum and in section 3 we will argue that externalities arise under this management situation. The policy seems to be in line with water management taking place in some countries where groundwater basins are exploited by a large number of independent pumpers extracting from a common groundwater resource. In this setting nothing is normally done when a drought hits and the aquifer is left to converge towards an open-access equilibrium. Hence, under this policy, the stock of water will automatically be restored to the pre-drought level after the event is over according to a zero marginal private net benefit condition and, thereby, scarcity rents are ignored.

The second policy also ignores the threat of drought and, thus, only deals with the drought when it hits. However, this policy recognizes that the water resource is an asset that should be managed over time taking account of intertemporal opportunity costs. Hence, we assume that nothing is done when the drought hits in the sense that the policy prior to the drought is still followed during the event. Therefore, the manager also in this case relies on automatically restoration through a zero marginal social net benefit condition when dealing with a drought. However, in actual policies less sophisticated water rationing policies are commonly used during a drought. Typically, these amount to keeping a constrained level of water from the date the drought hits until the aquifer has regained its normal size. Policies like these are used around the world, in particular in countries with an organized system of public water supply. The policies may take various forms such as cutting water supply during specific hours of the day or prohibiting certain usage of water (e.g. watering of private gardens).

The third policy takes account of both the threat of a drought and the intertemporal opportunity costs. It is the most sophisticated policy that may be implemented by prices set in accordance with marginal scarcity costs and this policy is identical to a full social optimum where the threat of the drought is fully incorporated. Water markets and pricing of water are now emerging and, in fact, this policy is common in many developed countries and regions around the world such as, for example, California.

Notation and assumptions

In the following we consider an aquifer of initial size on, x_0. It is assumed that the initial stock size is at a high level and x_0 is identical for all three management policies. The stock of water in the aquifer at date t is denoted x_t and the recharge rate (emanating from precipitation), $\bar{g}$, is assumed constant and stock independent. [9]. The social discount rate is denoted δ while the extraction from the aquifer at date t is endogenously determined and denoted, h_t. The industry

[9] Note that for fisheries the natural growth (recharge rate) is stock dependent implying that $G(x_t)$ and it is normally assumed that $G'(x_t) > 0$ for $x < x_{MSY}$ while $G(x_t) < 0$ for $x > x_{MSY}$, where x_{MSY} is the stock size correpsoding to maximum sustaianble yield.

gross benefit of water extraction, $U(h_t)$, is assumed to increase with extraction at a decreasing rate, i.e. $U(h_t)$ with $U'(h_t) > 0$ and $U''(h_t) \leq 0$. It is assumed that $U(h_t)$ captures the benefit of groundwater extraction in both the unregulated policy model and the models for the social optima. Thus, $U(h_t)$ may be interpreted as the revenue to farmers from extracting water for crop production.

The industry cost of water extraction, $C(x_t, h_t)$, is assumed linearly dependent on h_t such that $C(x_t, h_t) = c(x_t)h_t$. Furthermore, it is assumed that $c'(x_t) < 0$ and $c''(x_t) \geq 0$ such that a larger is the stock of water the lower is the marginal cost of extraction. This formulation of a cost function is common in the economic literature on groundwater, including papers on stochastic recharge. From this three facts may be noted in relation to this cost function. First, the assumption that a larger stock size of water implies a lower marginal cost of extraction has been confirmed in many empirical papers on groundwater extraction (see e.g. Koundouri, 2004 for an overview). Therefore we do not consider the special case where $c'(x_t) = 0$ in this paper. Second, the cost specification corresponds to an assumption of constant economics of scale of extraction.[10] Recently, Krishnamurthy (2016) have investigated the implications of non-constant economics of scale in extraction assuming stochastic recharge. However, when formally investigating whether a precautionary stock size should be built up prior to a drought (see section 5), an assumption about constant economics of scale in extraction is necessary. Therefore, we adopt the cost function with constant economics of scale of extraction in this paper. Third, we apply the same the cost function in all three models.

We consider a drought period of length, ε, starting at an uncertain date, τ. The event is assumed to hit only once in the foreseeable future (like a "hundred-year wave"). To represent this kind of uncertainty we apply an exponential function which is frequently used in problems dealing with unknown arrival of some event (see Barlow and Proschan, 1975). The essential assumption is that the conditional probability of the event happening, provided that it has not already happened,

[10] This may be seen from the fact that the average cost of extraction is equal to $c(x_t)$ and, therefore, this cost is independent of h_t.

remains constant as time proceeds. Hence, the length of the period prior to the drought does not matter and the distribution function is given by $F(t) = \Pr(\tau \leq t) = (1 - e^{-\lambda t})$ while the density function is equal to $f(t) = \Pr(\tau) = \lambda e^{-\lambda t}$. During the drought we assume that the precipitation to the aquifer vanishes and, hence, the recharge is equal to zero as for a non-renewable resource. After the drought period is over normal precipitation occurs and recharge rates are restored to the initial level given by $\bar{g}$. It is essential to note that we assume that the drought will happen sooner or later and the expected waiting time until the drought hits is equal to $1/\lambda$.

3. An open-access policy

Under open-access we let a representative private agent maximize the current time period net benefit without taking a resource restriction into account. We let extraction of groundwater, h_t, be the control variable and the maximization problem for period t is:

$$\underset{h_t}{Max}(U(h_t) - c(x_t)h_t) \tag{1}$$

From condition (1) the following first-order condition is reached:

$$U'(h_t) - c(x_t) = 0 \tag{2}$$

Thus, under open-access the marginal private net benefit is set equal to zero and, as pointed out by Provencher and Burt (1993), condition (2) reflects that two externalities exist when groundwater extraction is unregulated. First, a stock externality occurs because a resource restriction is disregarded. Second, a pumping cost externality arises because the private agent does not incorporate the effect of stock size on the cost when making decisions. Both externalities are associated with the stock size and in section 4 and 5, where we investigate the optimal stock sizes, these externalities are addressed.

Starting from a high initial stock size implying a low unit extraction cost, condition (2) implies that $h_t > \bar{g}$. Hence, as time passes, the stock of water will be driven down until the pre-drought steady-state stock size, x_∞, is reached. At x_∞, the extraction (h_∞) is equal to the recharge rate so that $h_\infty = \bar{g}$. This is a steady-state equilibrium and the stock size will remain at this level as long as the recharge is equal to $\bar{g}$.

When the drought hits, the stock of water starts to decrease since there is a net extraction of water. During a drought period of length, ε, the stock of water will decrease such that the stock size at date $\tau+\varepsilon$, $x_{\tau+\varepsilon}$, becomes:

$$x_{\tau+\varepsilon} = x_{\infty} - \int_{\tau}^{\tau+\varepsilon} h_t dt \qquad (3)$$

During the drought the extraction of water will also decrease, since the marginal extraction cost is increasing. This can be seen by utilizing condition (2) to obtain:

$$\dot{h} = -\frac{c'(x_t)h_t}{U''(h_t)} < 0 \qquad (4)$$

However, at date $\tau+\varepsilon$, the recharge is again positive and equal to, $\bar{g}$ and therefore the stock size of water starts to increase until the pre-drought steady-state stock size under open-access, x_{∞}, is again reached.

After the drought is over the extraction increases for a period until the steady state equilibrium stock size is obtained. Hence, for this period we have:

$$\dot{h} = \frac{c'(x_t)(\bar{g}-h_t)}{U''(h_t)} > 0 \qquad (5)$$

Thus, under open-access there is a built-in adjustment mechanism given by condition (2) to address the problems with a drought (a zero marginal private net benefit condition). Note that, water extraction is not sensitive to the size of the probability of a drought, but it is, however, sensitive to the length of the drought period. A longer drought period will lead to a lower extraction, a smaller water stock and an increased period of stock recovery.

4. A policy of optimal management ignoring the threat of drought

Under this policy we consider a management authority (a sole owner) with perpetual tenure managing the aquifer. It is assumed that the manager ignores the threat of a drought. The objective of the management authority is to maximize the present value of current and future net social benefit defined as the difference between (gross) benefits, *U*, and costs, C. When maximizing the present value of current and future net benefits h_t is the control variable and x_t is the state variable. Hence, the objective of the management authority is:

$$\max_{h_t} \int_0^\infty (U(h_t) - c(x_t)h_t)e^{-\delta t}dt \qquad (6)$$

subject to:

$$\dot{x}_t = \overline{g} - h_t \qquad (7)$$

$$x_0 = x(0) \qquad (8)$$

The current-value Hamiltonian corresponding to this problem is:

$$\mathrm{H} = U(h_t) - c(x_t)h_t + \gamma_t(\overline{g} - h_t) \qquad (9)$$

where γ_t denotes the adjoint variable.

The first-order necessary conditions for a maximum are:[11]

$$\frac{\partial \mathrm{H}}{\partial h_t} = U'(h_t) - c(x_t) - \gamma_t = 0 \qquad (10)$$

$$\frac{\partial \mathrm{H}}{\partial x_t} = -c'(x_t)h_t = \delta\gamma_t - \dot{\gamma}_t \qquad (11)$$

According to (10) the marginal social net benefit is set equal to zero. By comparing condition (10) and condition (11) with condition (2) we see the nature of both the stock externality and the pumping cost externality. First, in condition (10) an adjoint variable, γ_t, is included while this is not the case in condition (2). The adjoint variable captures the intertemporal opportunity cost of groundwater extraction. Hence the management authority takes the stock externality into account while this is not the case for a private agent operating under open-access. Second, under open-access a first-order condition for the state variable given by condition (11) does not exist. In condition (11) an important term is the marginal stock costs, $c'(x_t)h_t$, and, therefore, taking a first-order condition for the state variable into account corrects for the pumping cost externality.

By solving condition (11) for δ, taking the total differential of condition (10) with respect to time, using condition (7) and rearranging terms, we arrive at the following relationship:

[11] See Neher (190) for the optimality conditions for a current-value Hamiltonian.

$$\frac{U''\dot{h}_t - c'(x_t)\bar{g}}{U'(h_t) - c(x_t)} = \delta \qquad (12)$$

The steady-state solution, h^* and x^*(where $\dot{h}_t = \dot{x}_t = 0$ and $h^* = \bar{g}$), is characterized by:

$$-\frac{c'(x^*)\bar{g}}{U'(g) - c(x^*)} = \delta \qquad (13)$$

Condition (13) is known as the golden rule for optimal extraction of a renewable resource and this rule is carefully interpreted by Neher (1990).[12] Basically, condition (13) expresses that the optimal stock of water in a steady-state equilibrium arises where the marginal internal rate of return of keeping an extra volume unit of water is equal to the discount rate, δ. The marginal internal rate of return is the net return in terms of the gain of the lower extraction cost by keeping a marginally higher stock, $c'(x^*)\bar{g}$, shared by the net benefit forgone by not extracting an extra volume unit of water, i.e. $U'(\bar{g}) - c(x^*)$. The left hand side of condition (13) will later be referred to as the marginal stock cost effect.

Assuming that the initial value of stock size is larger than the steady-state equilibrium stock size ($x(0) > x^*$,) the optimal solution is to bring the stock of water asymptotically along a stable arm to the steady-state equilibrium stock size, x^*. The optimal strategy is to set the initial extraction rate at a value higher than $\bar{g}$ and then let h_t fall over time at a decreasing rate on an adjustment path towards the steady-state equilibrium value, h^*. Since the extraction rate is above the recharge rate, the size of the aquifer will also decrease along the adjustment path and approach the optimal steady-state equilibrium stock size, x^*. Hence, the stable arm for an initial stock size of water larger than the steady-state equilibrium stock size is characterized by $\dot{h}_t < 0, \dot{x}_t < 0$. In the steady-state equilibrium the adjoint variable, γ_t decreases at a rate equal to, $-\delta$, (i.e. $(\dot{\gamma}_t / \gamma_t) = -\delta$) implying a constant (current) implicit price of water.

When the drought hits we assume that the manager follows the optimal policy given by condition (13). Hence the stock of water will decline because there is no recharge and a positive extraction. The rate of extraction of water will decrease over time because the marginal extraction costs increase due to the decrease in stock size. After the drought is over the recharge again becomes equal to $\bar{g}$ implying that the stock size increase until x^* is reached. Due to the increase in stock size, the extraction cost will decrease leading to an increase in harvest. Thus, as under open-access, an automatic adjustment mechanism given by condition (10) exists when dealing with a drought (zero marginal social net benefit).

5. A policy of optimal management taking account of the threat of drought

As noted in section 2 this solution represents a full social optimum where the threat of the drought is incorporated. In solving this problem we must distinguish between three time periods: i) the period from until the drought hits (from $t = 0$ to $t = \tau$). ii), the period during the drought (from $t = \tau$ to $t = \tau + \varepsilon$).. iii), the period after the drought is over (from $t = \tau + \varepsilon$ to $t \to \infty$). As always in economics such a problem is solved by using backward induction. For all three time periods the control variable is denoted h_t while the state variable is denoted x_t .

The problem after the drought is over

At date $\tau + \varepsilon$ we are facing the same problem as discussed in section 4 and denoting the size of the aquifer at date $\tau + \varepsilon$ by $x_{\tau+\varepsilon}$, the maximum net present value of the future benefits evaluated at date $\tau + \varepsilon$ can be written as $S(h_t^*, x_t^*; x_{\tau+\varepsilon})$ which is the scrap value of the groundwater resource after the drought is over. Hence, from date, $\tau + \varepsilon$ and onwards, the aquifer is managed according to condition (12) with an initial groundwater stock size equal to $x_{\tau+\varepsilon}$.

The problem during the drought

Next, we move one step backwards in time and consider the optimization problem at date τ (shortly after the drought has hit). The optimization problem at this date may be formulated as:

$$\max_{h_t} \int_{\tau}^{\tau+\varepsilon} (U(h_t) - c(x_t)h_t)e^{-\delta t} dt \qquad (14)$$

subject to:

[12] Neher (1990) interpret the golden rule under the assumption that $U''(h_t)$ is constant. However, $U''(h_t)$ does not enter in condition (13) so this interpretation also holds for a non-constant $U''(h_t)$ as in this paper.

$$\dot{x}_t = -h_t \tag{15}$$

$$x_\tau = x(\tau) \tag{16}$$

Condition (15) states that the change in stock size is equal to the extraction reflecting that the aquifer is treated as a non-renewable resource during the drought while condition (16) is a restriction on the initial size of the stock of water at date τ .

The current-value Hamiltonian of the above problem is:

$$\mathrm{H} = U(h_t) - c(x_t)h_t - \phi_t h_t \tag{17}$$

where the adjoint variable is denoted ϕ_t. Condition (17) is to be maximized subject to the scrap value function, $S(h_t^*, x_t^*; x_{\tau+\varepsilon})$ covering the time periods after the drought is over.

The first order necessary conditions are:

$$\frac{\partial H}{\partial h_t} = U'(h_t) - c(x_t) - \phi_t = 0 \tag{18}$$

$$\frac{\partial \mathrm{H}}{\partial x_t} = -c'(x_t)h_t = \phi_t \delta - \dot{\phi}_t \tag{19}$$

Observe that discontinuities in both h_t and x_t at $t = \tau + \varepsilon$ the time where the drought is over occur because the recharge jumps from zero to $\overline{g}$. To show this let $\tau + \varepsilon^-$ be the instant of time immediately before the drought is over and $\tau + \varepsilon^+$ be the instant of time immediately after the drought is over. By using condition (10), condition (12), condition (18) and condition (20) we get that:

$$\frac{U''\dot{h}_{\tau+\varepsilon^-}}{\phi_{\tau+\varepsilon^-}} = \frac{U''\dot{h}_{\tau+\varepsilon^+} - c'(x_{\tau+\varepsilon^+})\overline{g}}{\phi_{\tau+\varepsilon^+}} \tag{20}$$

where $h_{\tau+\varepsilon^-}$ and $\phi_{\tau+\varepsilon^-}$ are the extraction and the adjoint variable at the instant of time before the drought is over while $h_{\tau+\varepsilon^+}, x_{\tau+\varepsilon^+}$ and $\phi_{\tau+\varepsilon^+}$ are the extraction, the stock size and the adjoint variable at the instant of time after the drought is over. From condition (20) and a transversality condition in Seierstad and Sydsæter (1987) we get that:

$$\phi_{\tau+\varepsilon^-} < \phi_{\tau+\varepsilon^+} = \frac{\partial S(h_t^*, x_t^*; x_{\tau+\varepsilon})}{\partial x_{\tau+\varepsilon^+}} \tag{21}$$

Thus, because $\phi_{\tau+\varepsilon^-} < \phi_{\tau+\varepsilon^+}$ there is a discontinuity in the adjoint variable at time $\tau+\varepsilon$ and by using condition (18) and condition (19) we get that discontinuities in h_t, and x_t will also have to exist.

By solving condition (19) for δ, taking the total differential of condition (18) with respect to time, substituting condition (15) into the expression and rearranging terms, we arrive at:

$$\frac{U''\dot{h}_t}{U'(h_t)-c(x_t)} = \delta \tag{22}$$

As mentioned above groundwater is treated as a non-renewable resource during the drought. Therefore, condition (22) represents Hotelling´s rule for optimal extraction of a non-renewable resource and condition (22) is identical to condition (13) with $\bar{g}=0$. From the fact that groundwater is a non-renewable resource during the drought it follows that x_t is non-increasing during the drought. Furthermore, because $c'(x_t) < 0$ it follows from condition (22) that $\dot{h} \le 0$ in condition (22) and therefore the extraction level will be non-increasing during the drought.

To simplify notation we now define the following maximum function:

$$F(x_\tau) = \max_{h_t}\left\{\int_0^{\varepsilon}\left[U(h_t)-c(x_t)h_t\right]e^{-\delta t}dt + S(h_t^*, x_t^*, x_{\tau+\varepsilon})e^{-\delta\varepsilon}\right\}, \tag{23}$$

subject to:

(15) and (16).

$F(x_t)$ can be interpreted as the maximum value of the net benefit arising from the date the drought hits and onwards and, thus, condition (23) represents a scrap value function of the groundwater resource at the time where the drought hits.

The problem at the initial moment of time

Finally, we move to the initial time period. The full ex ante problem to be considered in this period is as follows:

$$\max_{h_t}\int_0^{\infty}\lambda e^{-\lambda\tau}\left\{\int_0^{\tau}\left[U(h_t)-c(x_t)h_t\right]e^{-\delta t}dt + F(x_\tau)e^{-\delta\tau}\right\}d\tau \tag{24}$$

subject to:

$$\dot{x}_t = \overline{g} - h_t \qquad (25)$$

$$x_0 = x(0) \qquad (26)$$

By integrating the objective function by parts, it can be reformulated as:[13]

$$\max_{h_t} \int_0^{\infty} \left[U(h_t) - c(x_t)h_t + \lambda F(x_t) \right] e^{-(\delta+\lambda)t} dt \qquad (27)$$

The corresponding current-value Hamiltonian is:

$$\mathrm{H} = U(h_t) - c(x_t)h_t + \lambda F(x_t) + \mu_t(\overline{g} - h_t) \qquad (28)$$

where μ_t denotes the adjoint variable

The necessary first order conditions are:

$$\frac{\partial \mathrm{H}}{\partial h_t} = U'(h_t) - c(x_t) - \mu_t = 0 \qquad (29)$$

$$\frac{\partial \mathrm{H}}{\partial x_t} = -c'(x_t)h_t + \lambda F'(x_t) = (\delta + \lambda)\mu_t - \dot{\mu}_t \qquad (30)$$

Observe that a discontinuity in h_t and x_t at $t = \tau$ also exists because the recharge rate jumps form $\overline{g}$ to zero. To see this we let τ^- denote the instant of time just before the drought hits while τ^+ is the instant of time just after the drought has hit. By using the same procedure as for condition (20) we get:

$$\frac{U''\dot{h}_{\tau^+}}{\mu_{\tau^+}} = \frac{U''\dot{h}_{\tau^-} - c'(x_{\tau^-})\overline{g}}{\mu_{\tau^-}} \qquad (31)$$

where h_{τ^-}, x_{τ^-} and μ_{τ^-} are the extraction, the stock size and the adjoint variable at the instant of time before the drought hits and h_{τ^+} and μ_{τ^+} are the extraction and the adjoint variable at the instant of time after the drought has hit. By using condition (31) and the same transversality condition as above we find that:

$$\mu_{\tau^-} > \mu_{\tau^+} = F'(x_\tau) \qquad (32)$$

[13] See Dasgupta and Heal (1974) and Amundsen and Bjørndal (1999)

From condition (32) we have that $\mu_{\tau^-} > \mu_{\tau^+}$ and by using this fact in condition (29) and (30) we have that discontinuities in the extraction and the stock size arise at the time when the drought hits.

Condition (32) captures how the threat of a drought will affect the extraction level prior to the event. Clearly, transferring an extra unit of the groundwater resource from the time period prior to the drought to the time period during the drought will have a positive effect on the scrap value function ($F'(x_t) > 0$). By comparing this value with the marginal benefit of extraction before the drought hits, given by condition (29), we obtain that:

$$U'(h_\tau) - c(x_\tau) > F'(x_\tau) \tag{33}$$

The intuition for condition (33) is straightforward. At time $t = \tau$ (the time where the drought hits) there is a discontinuity in the recharge rate. If the extraction is to be constant this discontinuity will have to generate a decrease in the stock size equal to the missing recharge in the steady-state equilibrium. This decrease in x_t will lead to an increase in extraction costs because $c'(x_t) < 0$ and, therefore, the net benefit of water extraction will decrease during the drought. This decrease in the net benefit can be minimized by decreasing extraction at $t = \tau$. Therefore, $U'(h_t)$ will have to increase to compensate for the increase in extraction costs. Thus, we must that $U'(h_\tau) - c(x_\tau) > F'(x_\tau)$ as stated in condition (33).

By solving condition (30) for δ, taking the total differential of condition (29) with respect to time, using condition (25) and rearranging terms, we obtain the following condition:

$$\frac{U''\dot{h}_t - c'(x_t)\bar{g}}{U'(h_t) - c(x_t)} + \frac{\lambda\left[F'(x_t) - (U'(h_t) - c(x_t))\right]}{U'(h_t) - c(x_t)} = \delta \tag{34}$$

According to condition (34), extraction will decrease over time and the stock will decrease from the initial level, x_0, until a steady-state equilibrium extraction level, $\tilde{h}$, and a stock size, $\tilde{x}$, is reached.

The steady-state solution (with $\dot{h}_t = \dot{x}_t = 0$ and $\tilde{h} = \bar{g}$) is characterized by:

$$\frac{-c'(\tilde{x})\bar{g}}{U'(\bar{g})-c(\tilde{x})}+\frac{\lambda F'(\tilde{x})}{U'(\bar{g})-c(\tilde{x})}=\delta+\lambda \tag{35}$$

Condition (35) represents a modified golden rule for optimal management of a groundwater resource when the threat of a drought is incorporated. It essentially says that the rate of return from keeping a marginally higher stock size must be equal to the discount rate plus the failure rate, λ. The net benefit of keeping an extra unit of water in the stock is represented by the left hand side of condition (35) and consists of two elements. The first element is the stock cost effect described in condition (13). The second element is the expected future benefit of an extra unit of water after the date the drought has hit. These benefits accrue due to an "investment" of $\mu_t = (U'(\bar{g}) - c(\tilde{x}))$, which is equal to the net benefit foregone by not extracting a marginal unit of water shortly prior to the drought hits. By comparing the left hand side of condition (13) and condition (35) we see that the second term is an additional element arising when the threat of the drought is taken into account. Note that on the right hand side of condition (35) the failure rate, λ, acts like an increment to the discount rate.

It may be observed that if $\lambda = 0$ condition (35) reduces to condition (13). It can also be seen that provided $F'(\tilde{x}) = 0$ condition (35) reduces to:

$$-\frac{c'(\tilde{x})\bar{g}}{U'(\bar{g})-c(\tilde{x})}=\delta+\lambda \tag{36}$$

This expression corresponds to an expression in Dasgupta and Heal (1974) and Reed (1984), and condition (36) states that if the scrap-value function, $F(\cdot)$, is zero or unaffected by decisions taken prior to the drought, the failure rate, λ, acts like an increment to the discount rate. Hence, when $F'(\tilde{x}) = 0$ uncertainty implies heavier discounting and a lower stock of water in a steady-state equilibrium.

A precautionary stock size?

Next, we consider how $\tilde{x}$ is related to x^* (the steady-state stock size under policy ii)). This may be investigated by considering how the steady-state stock size responds to changes in λ. By taking the implicit derivative of condition (35) we obtain:

$$\frac{d\tilde{x}}{d\lambda} = \frac{\left[1 - \frac{F'(\tilde{x})}{U'(\bar{g}) - c(\tilde{x})}\right]}{\frac{\lambda\left[(U'(\bar{g}) - c(\tilde{x})F''(\tilde{x}) + F'(\tilde{x})c'(\tilde{x})\right] - \bar{g}\left[(U'(\tilde{x}) - c(\tilde{x})c''(\tilde{x}) + (c'(\tilde{x})^2\right]}{\left[U'(\bar{g}) - c(\tilde{x})\right]^2}} < 0 \quad (37)$$

Inspection of signs shows that the denominator in condition (37) is negative while the numerator is strictly positive (for $\varepsilon > 0, \lambda > 0$) due to the fact that $\left[U'(\tilde{h}) - c(\tilde{x})\right] > F'(\tilde{x})$ (condition (33)). Hence, an increase of the failure rate, λ, leads to a decrease in the optimal pre-drought steady-state equilibrium stock size and this implies that it is always optimal to keep a steady-state equilibrium stock size, $\tilde{x}$, that is smaller than or equal to x^*. Thus, it is not optimal to build up a precautionary stock of water to be used when the drought hits.

The result in condition (37) seems counter-intuitive but it can be explained by introducing some results from existing literature on renewable resources. Here we start by considering Sutinen (1981), Reed (1984) and Tsur and Zemel (1995) who all investigate the possibility of total extinction of a renewable resource. Sutinen (1981) analyze total extinction of a fish stock while Reed (1984) discusses a forest for which a risk of fire exists. Tsur and Zemel (1995) consider the possibility of total extinction of a groundwater resource due to an irreversible event. In all these papers it is shown that the optimal steady-state equilibrium stock size prior to the uncertain event will be lower when the possibility of extinction is included. The intuition for this result is that with a risk of extinction it is optimal to increase the extraction prior to the resource collapse to capture more net benefit before extinction. Thereby, the current stock size decreases and is moved towards the open-access level. This kind of behavior can be called an extinction effect and Sutinen (1981), Reed (1984) and Tsur and Zemel (1995) all formally show that this effect is identical to an increase in the discount rate. In our model the extinction effect is captured by λ in condition (35) and we reach the same conclusion as in Sutinen (1981), Reed (1984) and Tsur and Zemel (1995). However, Sutinen (1981), Reed (1984) and Tsur and Zemel (1995) consider a resource that becomes totally extinct. Contrary to this our problem allows for recovery of the resource after the uncertain event. Thus, our result in condition (37) is not directly comparable with the results in Sutinen (1981), Reed (1984) and Tsur and Zemel (1995).

More related to the analysis in our paper are the results in Amundsen and Bjørndal (1999) who study a fishery that is subject to a temporary collapse after which the fish stock recovers. According to Amundsen and Bjørndal (1999) two counteracting effects arise with a temporary collapse of a fish stock. First, as in Sutinen (1981), Reed (1984) and Tsur and Zemel (1995) an extinction effect exists leading a lower pre-collapse stock size and a higher pre-collapse extraction level. Second, a recovery effect arises based on a precautionary argument for keeping a larger pre-collapse stock size to reduce the recovery time after the collapse is over. In condition (35) the recovery effect is captured by the term $\frac{\lambda F'(\tilde{x})}{U'(\overline{g}) - c(\tilde{x})}$. Amundsen and Bjørndal (1999) obtain ambiguous results when considering the relative strength of the two effects for fisheries. It is therefore unclear whether a precautionary stock should be build up prior to a temporary collapse of a fish stock. By investigating a temporary drought as in this paper the same two effects as in Amundsen and Bjørndal (1999) arise, but we show that the extinction effect dominates the recovery effect. Thus, even though a recovery effect exists we obtain that $\tilde{x} \le x^*$. This can be explained by the fact that the recharge rate for groundwater is exogenous and cannot be affected by the decisions of a manager. In the fisheries case considered by Amundsen and Bjørndal (1999), the natural growth (the recharge rate) is stock dependent and keeping an extra unit of the stock size will lead to more than an one unit of the resource in the future due to the natural growth.[14] Thereby the recovery time is decreased. The strength of this effect may be so large that it dominates the extinction effect. In the groundwater case one unit of water not extracted now will only be equal to one unit of water in the future because the recharge is not affected by the extraction and thereby the stock size.

6. A numerical example

In this section the results of a numerical simulation of the three policies for groundwater management is presented. The three cases are: i) the open-access policy (Open-Access), ii) the optimal policy ignoring the threat of drought (Basic Solution) iii) the optimal policy including the

[14] This argument require that the marginal growth is negative implying that $x > x_{MSY}$. However, because $c'(x_t) < 0$ this requirements is fulfilled (see Jensen and Vestergaard, 2002).

threat of the drought (Optimal Solution). The assumed functional forms and parameter values are summarized in Appendix and the results of the simulations are presented in Figure I and Figure II.

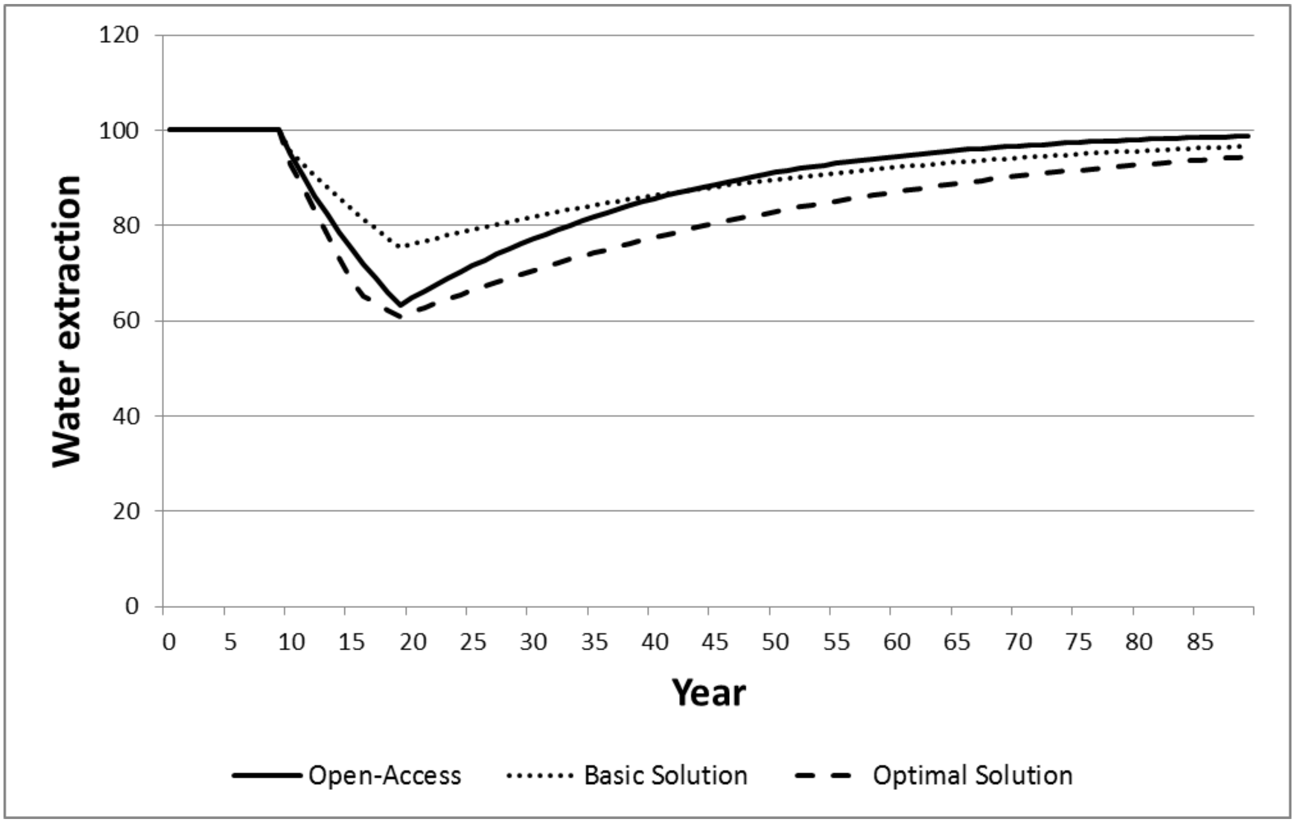

Figure I: Water extraction under various management policies.

Figure I shows that recovery time to normal extraction ($\bar{g}$), perhaps somewhat surprisingly, is fastest under Open Access. Compared to Basic Solution, Open Access involves a larger immediate decrease in the extraction but it increases more quickly. This is partly reflecting that the steady-state equilibrium stock size under Basic Solution is larger than under Open Access. The decrease in the water extraction under Optimal Solution is even more pronounced than under Open Access and, furthermore, water extraction under Optimal Solution recovers slowest of all policies. This is, partly, due to the fact that the stock of water prior to the drought is lower than under Basic Solution.

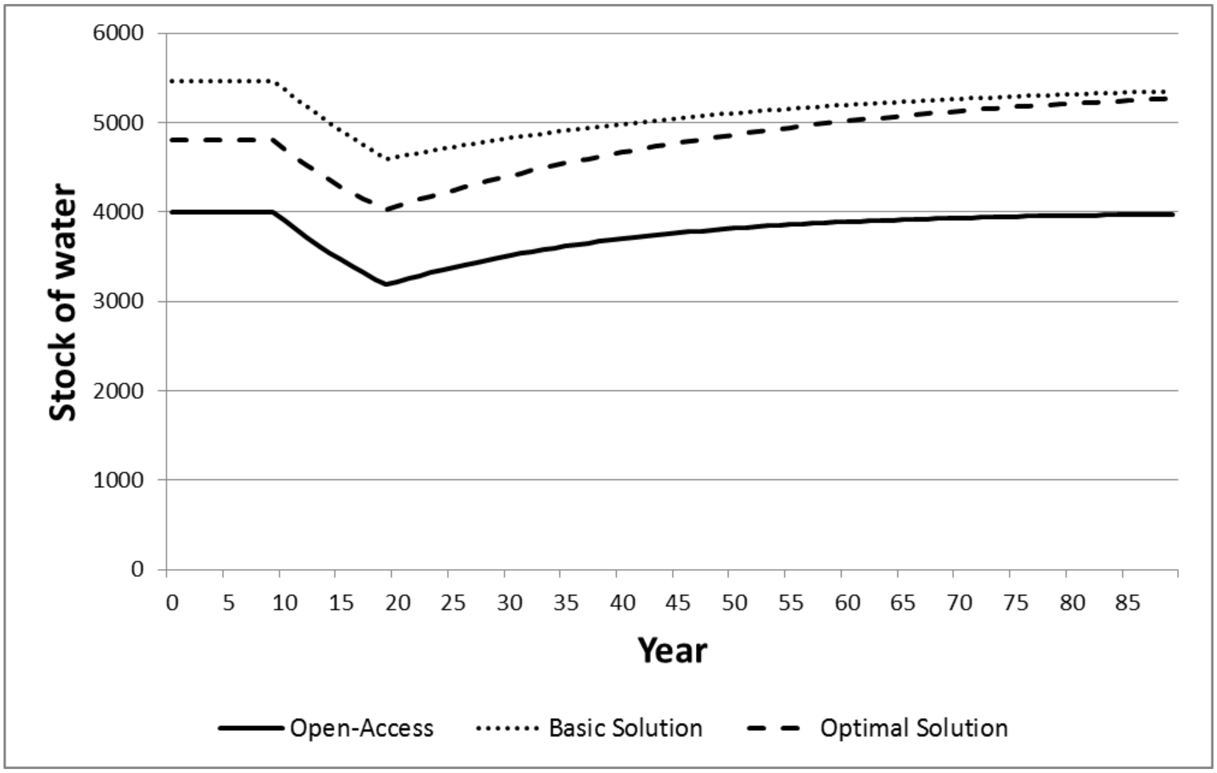

Figure II: Stock size of water under various management policies.

Comparing the various policies Figure II shows that the steady-state stock size of water is highest under Basic Solution, intermediate under Optimal Policy and lowest under Open Access. Naturally, the size of the extraction under steady-state is identical in the three policy cases and equal to the aquifer recharge, $\bar{g}$. The difference in stock size implies differences in the extraction costs of the opposite order and therefore the costs is lowest under the Basic Solution, intermediate under Optimal Policy and highest under Open Access.

7. Discussion

One important message from the analysis in this paper is that it is not necessarily true that keeping a steady-state equilibrium stock size of water with a low extraction cost is better than keeping a stock size with a high extraction cost. In fact, when a threat of a drought is included in an ex-ante maximization problem, the higher the probability of a drought (the failure rate, λ) the smaller is the optimal steady-state stock size and the larger is the extraction cost. For a sufficiently high drought probability the stock size of water may in fact be close to the open-access stock size implying the highest extraction cost. The reason for this result is that the failure rate acts like an increment to the discount rate leading to heavier discounting and, therefore, a smaller steady-state stock size (see condition (37)).

In the model in this paper we have assumed that the drought only hits once. This is of course an unrealistic assumption and it is natural to ask what will happen if droughts are allowed to be recurring events. An assumption of recurring droughts generally complicates matters a lot, and simplifying assumptions are needed in order to investigate the problem. Assume, as an example, that the stock of water is always fully recovered to the steady state level before the next drought hits (all assumed to be of equal length ε).[15] Since we are dealing with infinite time, the manager is, after each recovery period, faced with exactly the same optimization problem and thus arrives at exactly the same solution for the steady state equilibrium stock size in all time periods. Hence, the full ex ante optimization problem will be identical to the problem considered in section 5, except for the fact that the scrap value, $F(x_t)$, is replaced by an expected value of the maximum function for the future net benefit, $E\{F(x_t)\}$, that takes account of all expected future drought periods. Thus, we have:

$$\max_{h_t, x_t} \int_0^\infty \left[U(h_t) - c(x_t)h_t + \lambda E\{F(x_t)\}\right] e^{-(\delta+\lambda)t} dt \qquad (38)$$

subject to:

(25) and (26).

Denoting the optimal steady state equilibrium stock of water in this problem $\hat{x}$ the modified golden rule becomes:

$$-\frac{c'(\hat{x})\bar{g}}{U'(\bar{g}) - c(\hat{x})} + \frac{\lambda\left[E\{F'(\hat{x})\} - (U'(\bar{g}) - c(\hat{x}))\right]}{U'(\bar{g}) - c(\hat{x})} = \delta \qquad (39)$$

The structure of condition (38) is identical to condition (34) and, therefore, the steady-state equilibrium stock size will be smaller when the threat of the droughts is taken account of ($\hat{x} < x^*$). However, note that $F(x_t) > E\{F(x_t)\}$ because water is less abundant in a situation with

[15] This is not an innocent assumption since recovery time, as illustrated in Figure II, may be very long.

many drought periods and therefore the expected maximum future net benefit of water extraction during a drought must be smaller.

Two other assumptions behind the analysis in this paper are also useful to discuss. First, we assume constant economies of scale in extraction since the cost function is given as $c(x_t)h_t$. This assumption was used to proof that $\tilde{x} \leq x^*$. However, a natural question is whether this result generalizes to non-constant economics of scale in extraction. In relation to this Krishnamantha (2016) have recently shown that many results from the literature on stochastic recharge generalize to non-constant returns to scale in extraction and it seems that this conclusion also holds for the results in this paper. Within our model the cost assumption affect $\tilde{x}$ and x^* in an identical way and, therefore, it seems likely that $\tilde{x} \leq x^*$ also holds with non-constant economies of scale in extraction even though we have not proved this formally. Second, we only investigate a stock externality and a pumping cost externality in this paper but in reality other market failures arise for groundwater extraction. To investigate the implications of including other market failures consider, for example, the risk externality mentioned by Provencher and Burt (1993). The risk externality arises because risk-averse private extractors prefer a high groundwater stock size over a lower groundwater stock size due to a reduction in the risk. Taking account of such an externality will generally lead to higher optimal stock size.

8. Conclusions

Three management policies for an aquifer confronted with the threat of a drought have been considered; i) an open-access policy, ii) an optimal policy not taking the threat of a drought into account iii) an optimal policy taking account of the threat of drought. We show that under policy i) the water resource automatically adjusts towards the pre-drought level according to a zero marginal private net benefit condition. This policy is, however, not optimal since stock externalities and pumping cost externalities are not taken into account. Instead an aquifer should be managed over time as an asset according to the intertemporal opportunity costs in terms of scarcity/ resource rents. The other two policies include such intertemporal aspects.

In the first policy taking scarcity rents into account the threat of the drought is ignored and as under open-access the water resource is automatically adjusted towards the pre-drought level but now according to a zero marginal social net benefit condition. However, it is common to argue that the threat of a drought implies that a precautionary stock of water should be built-up prior to the drought to be drawn upon during the drought. A bit surprisingly we reach the opposite result since the optimal pre-drought stock size should be smaller than when the threat of the drought is ignored. This result implies that it is optimal to increase extraction of water during the period of normal precipitation before the drought starts. The intuition for this result is that two counteracting effects arise when the threat of a drought is taken into account. First, due to a temporary resource collapse current extraction is increased to capture a higher net benefit before the drought. This leads to a lower pre-drought stock size and this effect can be called an extinction effect. Second, because of an objective of fast resource recovery after the drought is over current extraction decreases and this can be called a recovery effect. In this paper we show that for a groundwater resource the extinction effect always dominates the recovery effect because the recharge is exogenous and this leads to a a lower pre-drought stock size. This result holds even if the drought period extends into infinity such that the aquifer becomes a non-renewable resource at the date the drought hits. Furthermore, even if the drought makes all or a part of the water stock unusable for extraction by, for example, triggering a salination process, the result will hold. Under simplifying assumptions, it can also be shown that the result will hold for recurring droughts. Hence, the result that the optimal steady-state stock of water should be smaller when the threat of the drought is included seems to be general and robust.

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Appendix

Figure I and Figure II are based on numerical examples using a gross benefit function and a cost function, respectively, equal to

$$U(h) = 2\sqrt{h}\,,\ C(h,x) = c(x)h = \frac{a}{x}h\,,\ \text{where } a = \text{a positive constant}$$

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$$h_t = h_\infty = \bar{g},\ \text{for } t \le \tau\,,\ h_t = \left(\frac{x_t}{a}\right)^2,\ \text{for } t > \tau$$

$$x_t = x_\infty = a\sqrt{h_\infty}\,,\ \text{for } t \le \tau\,,\ x_t = x_\infty - \sum_{\tau}^{t} h_t\,,\ \text{for } \tau < t \le \tau + \varepsilon\,,$$

$$x_t = x_{\tau+\varepsilon} + \sum_{\tau+\varepsilon}^{t} (\bar{g} - h_t)\,,\ \text{for } t > \tau + \varepsilon$$

Basic Solution

$$h_t = h^* = \bar{g},\ \text{for } t \le \tau$$

$$h_t = h_{t-1} - 2h_{t-1}\left[\delta - \frac{\delta a\sqrt{h_{t-1}}}{x_{t-1}}\right],\ \text{for } \tau < t \le \tau + \varepsilon$$

$$h_t = h_{t-1} - 2h_{t-1}\left[\delta - \frac{\delta a\sqrt{h_{t-1}}}{x_{t-1}} - \frac{a\bar{g}\sqrt{h_{t-1}}}{(x_{t-1})^2}\right],\ \text{for } t > \tau + \varepsilon$$

$$x_t = x^* = a\sqrt{h^*}(1 + \sqrt{1 + 4\sqrt{\bar{g}}/a\delta})/2\,,\quad \text{for} \quad t \le \tau\,,\quad x_t = x^* - \sum_{\tau}^{t} h_t\,,\quad \text{for} \quad \tau < t \le \tau + \varepsilon\,,$$

$$x_t = x_{\tau+\varepsilon} + \sum_{\tau+\varepsilon\tau}^{t} (\bar{g} - h_t)\,,\ \text{for } t > \tau + \varepsilon$$

Optimal Solution

$$h_t = \tilde{h} = \bar{g},\ \text{for } t \le \tau$$

$$h_t = h_{t-1} - 2h_{t-1}\left[\delta - \frac{\delta a\sqrt{h_{t-1}}}{x_{t-1}}\right], \text{ for } \tau < t \le \tau + \varepsilon$$

$$h_t = h_{t-1} - 2h_{t-1}\left[\delta - \frac{\delta a\sqrt{h_{t-1}}}{x_{t-1}} - \frac{a\bar{g}\sqrt{h_{t-1}}}{(x_{t-1})^2}\right], \text{ for } t > \tau + \varepsilon$$

$x_t = \tilde{x}$, implicitly given by

$$\frac{a\bar{g}\sqrt{\bar{g}}}{\tilde{x}(\tilde{x} - a\sqrt{\bar{g}})} + \lambda \frac{F'(\tilde{x})}{(1/\sqrt{\bar{g}}) - (a/\tilde{x})} = \delta + \lambda, \text{ for } t \le \tau,$$

$$x_t = \tilde{x} - \sum_{\tau}^{t} h_t, \text{ for } \tau < t \le \tau + \varepsilon, \; x_t = x_{\tau+\varepsilon} + \sum_{\tau+\varepsilon}^{t} (\bar{g} - h_t), \text{ for } t > \tau + \varepsilon$$

In the numerical examples we have used $a = 400, \delta = 0,05, \bar{g} = 100$ and $\varepsilon = 10$. From these values we get that $x_\infty = 4000, x^* = 5464$ and $\tilde{x} = 4800$.[16]

[16] It has not been possible to derive an analytic solution for $F(\tilde{x})$ and $F'(\tilde{x})$. For this case we have considered a steady-state value $\tilde{x} = 4800$. This value corresponds to $\lambda = 1$ and $F'(\tilde{x}) = 0,015764$. For this case we have a net benefit equal to $U'(\bar{g}) - c'(\tilde{x}) = (1/\sqrt{\bar{g}}) - (a/\tilde{x}) = 0,0791667 > F'(\tilde{x})$.