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Abstract

The economic concept of Returns-to-Scale (RTS) has been intensively studied in the context of Data Envelopment Analysis (DEA). The conventional DEA models that are used for RTS classification require well-defined and accurate data whereas in reality data are often imprecise, vague, uncertain or incomplete. The purpose of this paper is to estimate RTS of Decision Making Units (DMUs) in Imprecise DEA (IDEA) where the input and output data lie within bounded intervals. In the presence of interval data, we introduce six types of RTS involving increasing, decreasing, constant, non-increasing, non-decreasing and variable RTS. The situation for non-increasing (non-decreasing) RTS is then divided into two partitions; constant or decreasing (constant or increasing) RTS using sensitivity analysis. Additionally, the situation for variable RTS is split into three partitions consisting of constant, decreasing and increasing RTS using sensitivity analysis. Finally, we present the stability region of an observation while preserving its current RTS classification using the optimal values of a set of proposed DEA-based models.

Keywords: Returns-to-scale; Interval data; Data envelopment analysis.

JEL Classification C61 D24 D80

1. Introduction

Among the most important and highly discussed topics in the Data Envelopment Analysis (DEA) literature is the estimation of Returns-to-Scale (RTS) of individual Decision Making Units (DMUs) (i.e., observations). Since the paper on most productive scale size by Banker (1984) a series of papers have been devoted to various aspects of RTS classification in different types of DEA models (as we will briefly review in Section 1.1 below). In short, RTS classification poses two challenges. The most straightforward is to classify RTS of efficient DMUs which turn out to be closely related to the optimal solutions of the standard DEA models. The second is to classify the RTS of inefficient DMUs which further requires a relevant projection onto the efficient frontier of the production possibility set.

In the present paper we will consider yet another challenge that is related to data imprecision. In many empirical cases, data is often subject to substantial imprecision that seriously questions the relevance of well-defined "crisp" data as required by the standard DEA framework. In such cases, interval data seem more appropriate as recognized by two important strands of literature: Imprecise DEA (IDEA) and Fuzzy DEA (FDEA).

IDEA was originally presented by Cooper et al. (1999, 2001a, 2001b) when data sets include intervals, ordinal data and/or ratio bounds. Subsequently, a great deal of interest for interval DEA followed as briefly reviewed in Section 1.1 below.

The FDEA literature is recently surveyed in Emrouznejad et al. (2014) and Hatami-Marbini et al. (2011). Loosely speaking fuzzy data can be seen as generalized intervals, but many of the relevant methods build on so-called alpha-level based approach, which are in fact interval data.

When data is in the form of intervals it is no longer obvious what RTS means. In case of crisp data, Constant Returns-to-Scale (CRS) prevails when increasing the input use with a factor α increases output production with the same factor α. With inputs and outputs given by intervals we choose to apply a straightforward generalization of CRS in the sense that multiplying input intervals with a factor α, $[\alpha\ x^L, \alpha\ x^U]$, leads to multiplying output intervals with a factor α as well $[\alpha\ y^L, \alpha\ y^U]$. In terms of economics this corresponds to looking at imprecision as a relative uncertainty surrounding the value of a given variable, say, +/- 10% of the crisp value.

Mimicking the standard methods for RTS classification in the conventional (crisp) case we suggest using the approach of Wang et al. (2005) to determine the efficient frontier (in case of interval data). In particular we propose six RTS classes involving increasing, decreasing, constant, non-increasing, non-decreasing and variable RTS. We then use the sensitivity analysis to divide the case of non-increasing (non-decreasing) RTS into two partitions; constant or decreasing (constant or increasing) RTS and to split the situation for variable RTS into three partitions consisting of constant, decreasing and increasing RTS.

Finally, in the presence of bounded data we present the stability region of a given DMU while preserving its current RTS classification using the optimal values of a set of proposed DEA-based models as in the approach of Seiford and Zhu (1999b).

1.1. Related literature

Banker (1984) initially discussed how to identify RTS in the CCR formulation of DEA (Charnes et al. 1978) and the BCC model was proposed by Banker et al. (1984) under the assumption of variable RTS. Subsequently, Banker and Thrall (1992) and Zhu and Shen (1995), indicated some methods for estimating RTS when the BCC model encountered multiple optimal solutions. Banker et al. (1996) proposed an alternative algorithm to specify RTS when the CCR model has alternative solutions but their method has high complexity.

Färe et al. (1985, 1994) characterized types of RTS using the efficiency scores of DMUs. Golany and Yu (1997) proposed an algorithm to determine RTS of the efficient DMUs. Jahanshahloo et al. (2005) extended Golany and Yu (1997)'s method since their algorithm is limited to some cases. Seiford and Zhu (1999a, 1999b) proposed several DEA models for classifying RTS of DMUs in case of input and output orientation focusing on estimating stability regions of RTS.

In the non-radial models the classification of RTS is further complicated due to the multiple projections for each inefficient DMU. Sueyoshi and Sekitani (2007a) discussed RTS of the non–radial range-adjusted measure (RAM) model. They dealt with the RTS problem associated with non–radial DEA model by finding all the efficient DMUs that belong to the reference set. Subsequently, Sueyoshi and Sekitani (2007b) extended their model when the alternative optimal solutions occur in the reference set and supporting hyperplane. Fukuyama (2000) provided some mathematical properties of scale elasticity (SE) of the efficient and inefficient DMUs. Although Soleimani-damaneh and Mostafaee (2008) and Zhang (2008) claimed that Fukuyama (2000)'s results are incorrect, Fukuyama (2008) showed the correctness of the results.

Sueyoshi and Sekitani (2005) consider RTS in dynamic systems. Zarepisheh et al. (2006) introduced an algorithm to estimate the RTS of DMUs without chasing down alternative optimal solutions. Førsund et al. (2007) presented two approaches for the specification of RTS. The first approach radially projected DMUs on the frontier. They then used the efficiency score and its dual variables for specifying RTS associated with DMUs. The second approach used the intersection of hyperplanes that passes through the DMU for distinguishing its RTS. Soleimani-damaneh et al. (2009) studied the relation between the RTS and SE when there are the alternative solutions. Zarepisheh and Soleimani-damaneh (2009) estimated RTS on the left and on the right by means of the dual simplex method. Soleimani-damaneh et al. (2010) explored the relation between the RTS and SE in the presence of weight restrictions and the alternative solutions.

Sueyoshi and Goto (2011) considered DEA for the environmental assessment with the desirable and undesirable outputs. They then used a RAM model to estimate RTS and damages to scale (DTS) by means of desirable and undesirable outputs, respectively. Sueyoshi and Goto (2012) first defined the natural and managerial disposability concepts for environmental assessment. Next, they proposed non-radial model for distinguishing RTS and DTS of the natural and managerial disposability. Their method was applied to petroleum firms. Sueyoshi and Goto (2012) proposed radial and non-radial model for distinguishing RTS and DTS of the natural and managerial disposability. The authors applied their models to U.S. fossil fuel power. Sueyoshi and Goto (2013) first developed a technique to estimate RTS and DTS. Next, they applied their method to the US coal-fired power plants.

Witte and Marques (2011), and Soleimani-damaneh and Mostafaee (2009) proposed models for non-convex production possibility set (PPS). Soleimani-damaneh and Mostafaee (2009) studied the RTS of free disposal hull (FDH) model according to the summation of lambda as well as providing an algorithm to calculate the stability region of RTS classification. Witte and Marques (2011) measured the RTS for a FDH model according to the most imprecise scale size and applied their model for data from the Portuguese drinking water sector.

Following the IDEA papers by Cooper et al. (1999, 2001a, 2001b), Entani et al. (2002) proposed a DEA model with interval and fuzzy data to measure the interval efficiencies of DMUs from the optimistic and pessimistic viewpoints. Despotis and Smirlis (2002) proposed a pair of DEA models to calculate the upper and lower efficiency when input and output data vary in intervals. They defined 2n PPS for evaluating n DMUs from the best and worst viewpoints. Wang et al. (2005) modified Despotis and Smirlis (2002)'s models by introducing a unified PPS in conjunction with n DMUs under assessment. Lee et al. (2002) presented a non-linear additive imprecise DEA (IDEA) model that was converted to the linear programming model. Kao (2006) expressed the imprecise DEA problem as a bi-level mathematical programming model when input and output data are characterized by the intervals. Park (2010) researched the relationship between primal and dual with imprecise data. Emrouznejad et al. (2011) developed the imprecise DEA models with interval data to estimate overall profit efficiency. The authors obtained the upper and lower bounds of the overall profit efficiency for DMUs and proposed a classifying the scores. Emrouznejad et al. (2012) developed general non-parametric corporate performance (GNCP) model and multiplicative non-parametric corporate performance (MNCP) model with interval ratio data. Shokouhi et al. (2010, 2014) used the robust optimization concept to propose two different robust data envelopment analysis (RDEA) models for dealing with data uncertainty.

The fuzzy DEA methods (as recently surveyed in Emrouznejad et al., 2014; Hatami-Marbini et al., 2011), can be classified into six main categories: (1) the tolerance approach, see, e.g. Sengupta (1992), (2) the α-level based approach, see, e.g. Triantis and Girod (1998), Kao and Liu (2000), Hatami-Marbini et al. (2010), (3) the fuzzy ranking approach, see, e.g. Guo and Tanaka (2001), (4) the possibility approach, see, e.g. Lertworasirikul et al. (2003), Tavana et al. (2012), (5) the fuzzy arithmetic, see, e.g. Wang et al. (2009), and (6) the fuzzy random/type-2, see, e.g. Qin and Liu (2010).

1.2 Organization

The rest of the paper is organized as follows: In section 2, we first review the conventional CCR model and CCR RTS method. Then, we present an overview of the imprecise CCR model and the sensitivity analysis method for RTS estimation. In section 3, we present a simple example to graphically show the concept of RTS with interval inputs and interval outputs. In section 4, we develop a method to estimate RTS of DMUs with interval data followed by a discussion on stability of the RTS. We present an empirical example to illustrate the proposed method in Section 6. Finally, we provide some concluding remarks and suggestions for future research in Section 7.

2. Preliminaries

Data Envelopment Analysis (DEA) is a non-parametric method for evaluating the technical efficiency of a set of Decision-Making Units (DMUs). In this section, we present the classic CCR model with precise data and review a central characterization result for determining Returns-to-Scale (RTS) of efficient DMUs. We then briefly review the imprecise DEA model with interval data and the sensitivity analysis method for RTS estimation.

2.1. CCR model

The problem of evaluating of the performance of DMU_o can be formulated by linear programming. Suppose that we have n DMUs where each DMU_j, $j=1,\ldots,n,$ produces s outputs y_{rj} $(r=1,\ldots,s)$, using m inputs x_{ij} $(i=1,\ldots,m)$. Charnes et al. (1978) present the following *envelopment form* of the "CCR model" for measuring the radial input-efficiency of DMU_o:

$$\begin{aligned}
\min \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \qquad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \qquad r = 1, \dots, s, \\
& \lambda_j \ge 0 \;, \qquad j = 1, \dots, n.
\end{aligned} \tag{1}$$

where λ_j (j=1,…,n) is the weight placed on each DMU for making up the efficient facet of DMU_j. A DMU is called "CCR efficient" if and only if the objective function of model (1) is equal to 1, i.e., if $\theta^* = 1$, otherwise, it is called "CCR inefficient".

Within the DEA framework, the economic concept of RTS has received a great deal of attention (see, e.g., Banker, 1984; Banker et al. 1984). On the basis of model (1), Banker and Thrall (1992) put forward the following theorem characterizing RTS of given efficient DMUs:

Theorem 1. ***(Banker and Thrall, 1992)*** Suppose that $(\hat{x}_o, \hat{y}_o)$ is CCR efficient and λ_j^* is an optimal solution of model (1).

(I) Increasing returns-to-scale (IRS) prevail at $(\hat{x}_o, \hat{y}_o)$ if $\sum_{j=1}^{n} \lambda_j^* < 1$ for all alternate optimal solutions.

(II) Decreasing returns-to-scale (DRS) prevail at $(\hat{x}_o, \hat{y}_o)$ if $\sum_{j=1}^{n} \lambda_j^* > 1$ for all alternate optimal solutions.

(III) Constant returns-to-scale (CRS) prevail at $(\hat{x}_o, \hat{y}_o)$ if $\sum_{j=1}^{n} \lambda_j^* = 1$ in some optimal solutions.

It should be emphasized that Theorem 1 only holds for efficient DMUs. Therefore, the optimal value of θ^* from model (1) is recognized as a prerequisite for executing the RTS analysis. Banker at al. (1996) further developed a method to omit the examining all the alternate optima. The RTS classification of an inefficient unit can be identified via its projection onto the efficient frontier. Thereby, when studying the inefficient units, the input- and output-oriented CCR models may lead to different RTS classifications. In this paper we shall focus on *the input oriented version* of the CCR model for identifying the RTS characterizations.

2.2. Imprecise CCR model

The conventional DEA models such as CCR and BCC require the precise measurement of inputs and outputs. However, in certain situations data are naturally represented by intervals expressing ranges of relevant values, due to various forms of data imprecision. Interval data were initially applied in DEA by Cooper et al. (1999, 2001a, 2001b) and the application of the interval data in distinct DEA models was subsequently adopted by many researchers. We here briefly review the well-known imprecise (multiplier) DEA approach proposed by Despotis and Smirlis (2002).

Let n DMUs each produce s interval outputs $y_{rj} \in \left[y_{rj}^L, y_{rj}^U \right]$ (r=1,…,s) using m interval inputs $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ (i=1,…,m) where y_{rj}^L, y_{rj}^U, x_{ij}^L and x_{ij}^U, are strictly positive. Despotis and Smirlis (2002) proposed a pair of multiplier CCR models to compute the upper and lower bound of efficiency, here presented in their envelopment forms:

$$
\begin{aligned}
&\theta^U = \min\ \theta \\
&s.t.\quad \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j x_{ij}^U + \lambda_o x_{io}^L \le \theta x_{io}^L, && i = 1,\ldots,m, \\
&\qquad\ \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j y_{rj}^L + \lambda_o y_{ro}^U \ge y_{ro}^U, && r = 1,\ldots,s, \\
&\qquad\ \lambda_j \ge 0\ , && j = 1,\ldots,n.
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&\theta^L = \min\ \theta \\
&s.t.\quad \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j x_{ij}^L + \lambda_o x_{io}^U \le \theta x_{io}^U, && i = 1,\ldots,m, \\
&\qquad\ \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j y_{rj}^U + \lambda_o y_{ro}^L \ge y_{ro}^L, && r = 1,\ldots,s, \\
&\qquad\ \lambda_j \ge 0\ , && j = 1,\ldots,n.
\end{aligned}
\tag{3}
$$

θ^U and θ^L are called the efficiency score of DMU_o in the best and worst conditions respectively. As noticed by Wang et al. (2005) it is somewhat problematic that using (2) and (3) in effect implies that upper and lower efficiency scores are computed relative to different frontiers for the same DMU and hence introduces comparability issues. Consequently, Wang et al. (2005) developed a pair of multiplier DEA models using the same production possibility set to obtain the interval efficiency for each DMU. We apply the approach of Wang et al. (2005) to the envelopment CCR model resulting in the following models:

$$
\begin{aligned}
\bar{\theta}^{U} = \min \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{L} \le \theta x_{io}^{L}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{U} \ge y_{ro}^{U}, \quad r = 1, \ldots, s, \\
& \lambda_j \ge 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\bar{\theta}^{L} = \min \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{L} \le \theta x_{io}^{U}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{U} \ge y_{ro}^{L}, \quad r = 1, \ldots, s, \\
& \lambda_j \ge 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{5}
$$

where $\bar{\theta}^{L}$ and $\bar{\theta}^{U}$ are the lower and upper bounds of efficiency for DMU_o.

The only difference between the BCC and CCR models is the inclusion of the convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ in both models (4) and (5).

A DMU_o is called *efficient* if and only if the objective function of model (4) is equal to 1, otherwise, it is called *inefficient*.

2.3. Sensitivity of RTS classifications in the standard DEA model

The sensitivity of RTS classifications is an interesting and challenging research topic in the DEA literature. The estimation of RTS in DEA often takes into account the proportional change in all the outputs of DMU_o derived from a proportional change in all its inputs. In the standard input-oriented DEA model the RTS classifications of a DMU is not changed by variation of input levels unless the DMU is on the efficient frontier. Seiford and Zhu (1999b) present a sensitivity analysis framework by means of several linear programming models for exploring the stability of RTS classifications when the output levels are perturbed. In addition to the identification of the stability region for the RTS classifications (constant, increasing or decreasing returns-to-scale), the authors determined the RTS classification by the optimal values to a series of CCR-based models.

Within the input-oriented model, it is noticeable that if IRS prevails for a DMU, then its IRS cannot vary with decreases in outputs whereas if DRS prevail for a DMU, then its DRS remains unchanged with augmentations in outputs unless the DMU gets to the CCR frontier.

The following pair of programs can be used to detect the RTS classifications of DMU_o (Seiford and Zhu, 1999b):

$$\begin{aligned} (\tau_o^*)^{-1} &= \min \sum_{j\in E_o} \widehat{\lambda_j} \\ s.t. \quad & \sum_{j\in E_o} \widehat{\lambda_j} x_{ij} \le \theta^* x_{io}, \quad i=1,\dots,m, \\ & \sum_{j\in E_o} \widehat{\lambda_j} y_{rj} \ge y_{ro}, \quad r=1,\dots,s, \\ & \widehat{\lambda_j} \ge 0, \quad j \in E_o. \end{aligned} \tag{6}$$

$$\begin{aligned} (\sigma_o^*)^{-1} &= \max \sum_{j\in E_o} \widehat{\lambda_j} \\ s.t. \quad & \sum_{j\in E_o} \widehat{\lambda_j} x_{ij} \le \theta^* x_{io}, \quad i=1,\dots,m, \\ & \sum_{j\in E_o} \widehat{\lambda_j} y_{rj} \ge y_{ro}, \quad r=1,\dots,s, \\ & \widehat{\lambda_j} \ge 0, \quad j \in E_o. \end{aligned} \tag{7}$$

where θ^* is the optimal value of the standard CCR model (1) and $E_o =$ {all efficient DMUs in model (1)}. The following theorem demonstrates how to determine the RTS classification using the above-defined models.

Theorem 2. (Seiford and Zhu, 1999b) Suppose that τ_o^* and σ_o^* are the optimal values of models (6) and (7), respectively. The following conditions specify the RTS of DMU_0

(I) CRS prevail for DMU_o if and only if $\sigma_o^* \le 1 \le \tau_o^*$,

(II) IRS prevail for DMU_o if and only if $\sigma_o^* > 1$,

(III) DRS prevail for DMU_o if and only if $\tau_o^* < 1$.

To implement the sensitivity analysis of RTS classification (CRS, IRS and DRS), the three following theorems were developed by Seiford and Zhu (1999b) to deal with situations when output perturbations occur in DMU_o under the input-oriented DEA model.

Theorem 3. (Seiford and Zhu, 1999b) Let DMU_o exhibit CRS. Then, its CRS classification remains unchanged for $\gamma \in R^{CRS} = \left\{ \gamma \middle| \min\{1, \sigma_o^*\} \le \gamma \le \max\{1, \tau_o^*\} \right\}$ where γ indicates a proportional variation of all outputs, $\tilde{y}_{ro} = \gamma y_{ro} (r = 1, \ldots, s)$ and τ_o^* and σ_o^* are obtained by models (6) and (7) respectively.

Theorem 4. (Seiford and Zhu, 1999b) Let DMU_o exhibit IRS. Then, its IRS classification remains unchanged for $\alpha \in R^{IRS} = \{\alpha \mid 1 \le \alpha < \sigma_o^*\}$ where α is a proportional augmentation of all outputs $\tilde{y}_{ro} = \alpha y_{ro} (r = 1, \ldots, s)$ and σ_o^* is calculated by model (7).

Theorem 5. (Seiford and Zhu, 1999b) Let DMU_o exhibit DRS. Then, its DRS classification remains unchanged for $\beta \in R^{DRS} = \{\beta \mid \tau_o^* < \beta \le 1\}$ where β indicates a proportional variation of all outputs $\tilde{y}_{ro} = \beta y_{ro} (r = 1, \ldots, s)$ and τ_o^* is the optimal objective function of model (6).

3. Motivating Example and a few Preliminary Observations

Let us graphically display the shortcomings of the Despotis and Smirlis (2002) approach using a simple example that consists of six DMUs, marked as A, B, C, D, E and F in Fig. 1 where these DMUs respectively produce one interval output [1, 2], [1, 3], [4, 6], [6.5, 7], [5, 6.5] and [1, 6.5] using one interval input [2.5, 4], [3, 6], [6, 10], [7.5, 9], [10, 12] and [7, 10]. Notice that the coordinates of each point are represented in the order (*x*, *y*). A rectangle constructed by the interval data indicates the DMUs as shown in Fig. 1.

-------Insert Fig.1 here-------

When calculating upper bound efficiency of the six DMUs under a Variable Returns-to-Scale (VRS) assumption each evaluated DMU autonomously uses its best situation and the worst situation for other DMUs to build the six different production frontiers.

For instance, focus on DMU_A and DMU_C. When evaluating the upper efficiency bound of DMU_A, the points *k* and *t* in Fig. 1 as the best and worst situations are considered to form a piecewise DEA frontier *wkp* (black dashed line) that problematically disregards the *free disposability* and *convex hull* assumptions of standard DEA. Likewise, in assessing DMU_C an alternative production frontier as the piecewise linear form *uthp* (blue dotted line) is established using the point *h* as the best setting of DMU_C and the points *t* and *p* as the worst position of DMU_A and DMU_D, respectively. Consequently, the production possibility set based on the Despotis and Smirlis (2002) approach is not only not unique (equals to twice the number of DMUs), but it also violates the fundamental DEA axioms, see e.g., Banker et al. (1984).

To deal with these drawbacks, we define a unique piecewise BCC frontier based on Wang et al. (2005)'s approach mentioned in Section 2. The piecewise frontier here consists of *wk*, *kq*, *qh*, *hg* and *gb* (red dashed line) that starts at the vertical line (support at *k*) and ends through a horizontal line at *gb* (support at *g*), as shown in Fig 1. To build the frontier, we use the best situation of all DMUs (i.e., the smallest and greatest value of inputs and outputs for each DMU), in other words, the data set {(2.5,2), (3,3), (6,6), (7.5,7), (10,6.5), (7,6.5)} is used to form the production frontier. We take this piecewise frontier into account to determine the RTS classification of each DMU since this unique frontier envelops all the imprecise observations as tightly as possible and avoids the above-mentioned shortcomings. The line segments *wk*, *kq*, *qh*, *hg* and *gf* present IRS, IRS, CRS, DRS, DRS. Ray *oqh* (black dash-dotted line) is the CCR efficient frontier giving the case of CRS. If a DMU is on or projected onto the CCR efficient frontier, then this DMU exhibits the condition for CRS. When we encounter two distinct RTSs at the intersections such as *q* and *h*, we give priority to CRS-viz., on the line segment *kq* IRS prevail to the left of *q*, whereas on the line segment *hg*, DRS prevail to the right of *h*.

Unless a DMU lies on the production frontier the interpretation of RTS is not straightforward. That is to say, the RTS for the inefficient DMUs can be identified after applying the projection. It should be noted that input- and output-oriented models may lead to different RTS classifications for the same DMU (Golany and Yu, 1994). We therefore limit our study to the input-oriented radial model (a similar approach can be developed for output efficiency with the straightforward changes).

Now, consider the interval data for DMU_A that is presented by a rectangle. The left width of the rectangle of DMU_A is on the line segments *wk* where IRS prevails. The input-oriented projection (input-reduction) of the residual points of the DMU_A will be located on the line segment *wk* where IRS prevails. Therefore, the RTS classification for DMU_A is IRS where the input and output varies in a range.

As a result, two points A= (x_{io}^L, y_{ro}^L) and B= (x_{io}^L, y_{ro}^U) of each DMU enable us to exhibit its RTS classification. For instance DMU_A exhibits IRS because IRS prevails at the two points A= (2.5,1) and B=(2.5,2). DMU_B exhibits IRS and CRS simultaneously and it is consequently called Non-Decreasing Returns-to-Scale (NDRS). In other words, IRS prevails when the input and output lie within [3, 6] and [1, 3), respectively, and CRS prevails when the input lies within [3, 6] input and the output is 3. We can obtain the same result if the two points A=(3,1) and B=(3,3) of DMU_B are used for RTS classification. It is obvious that there is no point of DMU_B between points A and B that exhibits DRS. In fact, we have the following results.

Observation 1: If DMU_o exhibits IRS (that is, both points A and B exhibit IRS), increases in outputs below the CRS output level cannot change its IRS classification.

Proof. Let a DMU be associated with its (optimistic) point B. Thus, let $DMU_o = (x_o, y_o)$ exhibit IRS. By Theorem 4 (adapted from Seiford and Zhu, 1999b) the IRS classification remains unchanged for $\alpha \le \sigma_o^*$ as determined in model (7). Due to the inputs change does not alter the RTS, let us evaluate $(\sigma_o^* x_{io}, \sigma_o^* y_{ro})$ using model (7) as presented below:

$$(\bar{\sigma}_o)^{-1} = Max \sum_{j \in E_o} \hat{\lambda}_j^*$$

$$\sum_{j \in E_o} \hat{\lambda}_j^* x_{ij} \le \theta_o^* \left(\sigma_o^* x_{io}\right), \qquad i = 1,\ldots,m,$$

$$\sum_{j \in E_o} \hat{\lambda}_j^* y_{rj} \ge \sigma_o^* y_{ro}, \qquad r = 1,\ldots,s,$$

$$\hat{\lambda}_j^* \ge 0 \qquad j \in E_o.$$

where θ^* is the optimal objective function value of model (4) when evaluating $(\sigma_o^* x_{io}, \sigma_o^* y_{ro})$. Obviously, $\left(\hat{\lambda}_j^* / \sigma_o^*\right) \ge 0, j \in E_o$ is the feasible solution for model (7) when evaluating (x_{io}, y_{ro}).

Due to $\left(\sum_{j\in E_o}\hat{\lambda}_j^*\big/\sigma_o^*\right)\le 1\big/\sigma_o^* < 1$ we have $\left(\sum_{j\in E_o}\hat{\lambda}_j^*\right)\le 1 < \sigma_o^*$. We already know $(x_{io}, \sigma_o^* y_{ro})$ is not IRS, therefore, $\sum_{j\in E_o}\hat{\lambda}_j^* = 1$ and the RTS of DMU_o is CRS. ■

The rectangles associated with DMU_C and DMU_D exhibit CRS and DRS, respectively. DMU_E reveals DRS and CRS at the same time and it is consequently called Non-Increasing Returns-to-Scale (NIRS). Put differently, DRS prevails when the input and output lie within [10, 12] and (6,6.5], respectively, and CRS prevails when the input and output lies within [10,12] and [5,6], respectively.

Observation 2: If DMU_o exhibits CRS (that is, both points A and B exhibit CRS), increases (decreases) in outputs cannot change its CRS classification unless for point B, DRS (for point A, IRS) prevails.

Proof. As in the case of Observation 1 above we can now use Theorem 3 (adapted from Seiford and Zhu, 1999b) with respect to points B and A respectively. ■

The rectangle associated with DMU_F exhibits IRS, CRS and DRS at the same time. In other words, DRS prevails when the input and output lie within [7, 10] and (6,6.5], respectively, CRS prevails when the input and output lie within [7, 10] and [3,6], respectively and IRS prevails when the input and output lie within [7, 10] and [1,3), respectively. Therefore, the RTS classification of DMU is called Variable Returns-to-Scale (VRS). We obtain this result from the two points A=(7, 1) and B=(10, 6.5) which exhibit IRS and DRS, respectively. The DMU is thus classified into a VRS group according to Observation 3.

Observation 3: If DMU_o exhibits IRS (that is, point A exhibits IRS), and increases in outputs change its IRS to DRS (that is, point B exhibits DRS), then there is at least one potential output level for which CRS prevails. In other words, when $\text{DMU}_o = (x_o, y_o)$ exhibits IRS, increase the outputs of $\text{DMU}_o = (x_o, y_o)$ by $\gamma > 1$ such that $\text{DMU}'_o = (x_o, \gamma y_o)$ exhibits DRS. Then $\text{DMU}_o'' = (x_o, \chi y_o)$, $1 < \chi < \gamma$ exhibits CRS.

Proof. The proof is straightforward according to Observation 1 (omitted). ■

4. RTS classification with interval data

In order to determine the RTS classification of a given DMU characterized by the two (extreme) points A= (x_{io}^L, y_{ro}^L) and B= (x_{io}^L, y_{ro}^U) as discussed above, we first present models (8) and (9) to calculate their relative efficiencies as follows:

$$
\begin{aligned}
\min \quad & \theta_o^A \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^L \le \theta_o^A x_{io}^L, && i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^U \ge y_{ro}^L, && r = 1, \ldots, s, \\
& \lambda_j \ge 0, && j = 1, \ldots, n.
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\min \quad & \theta_o^B \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^L \le \theta_o^B x_{io}^L, && i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^U \ge y_{ro}^U, && r = 1, \ldots, s, \\
& \lambda_j \ge 0, && j = 1, \ldots, n.
\end{aligned} \tag{9}
$$

Adapting the Seiford and Zhu (1999b) framework for points A= (x_{io}^L, y_{ro}^L) and B= (x_{io}^L, y_{ro}^U) results in the following pair of models:

<u>Models (10) for point A</u>

$$
\begin{aligned}
(\sigma_o^A)^{-1} = \max & \sum_{j \in E_o} \widehat{\lambda}_j \\
s.t. \quad & \sum_{j \in E_o} \widehat{\lambda}_j x_{ij}^L \le \theta_o^{A*} x_{io}^L, && i = 1, \ldots, m, \\
& \sum_{j \in E_o} \widehat{\lambda}_j y_{rj}^U \ge y_{ro}^L, && r = 1, \ldots, s, \\
& \widehat{\lambda}_j \ge 0, && j \in E_o.
\end{aligned}
$$

<u>Models (11) for point B</u>

$$
\begin{aligned}
(\sigma_o^B)^{-1} = \max & \sum_{j \in E_o} \widehat{\lambda}_j \\
s.t. \quad & \sum_{j \in E_o} \widehat{\lambda}_j x_{ij}^L \le \theta_o^{B*} x_{io}^L, && i = 1, \ldots, m, \\
& \sum_{j \in E_o} \widehat{\lambda}_j y_{rj}^U \ge y_{ro}^U, && r = 1, \ldots, s, \\
& \widehat{\lambda}_j \ge 0, && j \in E_o.
\end{aligned}
$$

$$(\tau_o^A)^{-1} = \min \sum_{j \in E_o} \widehat{\lambda_j} \qquad\qquad (\tau_o^B)^{-1} = \min \sum_{j \in E_o} \widehat{\lambda_j}$$

$$\begin{aligned} s.t. \quad & \sum_{j \in E_o} \widehat{\lambda_j} x_{ij}^L \le \theta_o^{A*} x_{io}^L, && i = 1,\dots,m, \\ & \sum_{j \in E_o} \widehat{\lambda_j} y_{rj}^U \ge y_{ro}^L, && r = 1,\dots,s, \\ & \widehat{\lambda_j} \ge 0 \ , && j \in E_o. \end{aligned} \qquad \begin{aligned} s.t. \quad & \sum_{j \in E_o} \widehat{\lambda_j} x_{ij}^L \le \theta_o^{B*} x_{io}^L, && i = 1,\dots,m, \\ & \sum_{j \in E_o} \widehat{\lambda_j} y_{rj}^U \ge y_{ro}^U, && r = 1,\dots,s, \\ & \widehat{\lambda_j} \ge 0 \ , && j \in E_o. \end{aligned}$$

where θ_o^{A*} and θ_o^{B*} are the optimal solution of models (8) and (9), respectively and $E_o = \{$all efficient DMUs in models (8) and (9)$\}$. Therefore, we customize Theorem 2 to propose the following definitions:

(I) A (or B) exhibits IRS if and only if $\sigma_o^{A*} > 1 \ (\sigma_o^{B*} > 1)$.

(II) A (or B) exhibits DRS if and only if $\tau_o^{A*} < 1 \ (\tau_o^{B*} < 1)$.

(III) A (or B) exhibits CRS if and only if $\sigma_o^{*A} \le 1 \le \tau_o^{*A} (\sigma_o^{*B} \le 1 \le \tau_o^{*B})$.

Let us return to the earlier example in Section 3 to illustrate the above step of the proposed method. To identify RTS of A= (2.5, 1) and B=(2.5, 2) associated with DMU_A we use the optimal solution of models (10) and (11) which are $\sigma_o^{A*} = 3 > 1 \ (\sigma_o^{B*} = 1.5 > 1)$. Thereby, A and B exhibit IRS. Table 1 is summarized the findings for all the DMUs.

-------Insert Table 1 here-------

The following conditions determine the RTS classification of DMU_o in terms of its RTS estimation for the two points A and B:

Con. 1. DMU_o exhibits IRS iff B exhibits IRS.

Con. 2. DMU_o exhibits DRS iff A exhibits DRS.

Con. 3. DMU_o exhibits CRS iff A and B exhibit CRS.

Con. 4. DMU_o exhibits NDRS iff A and B exhibit IRS and CRS, respectively.

Con. 5. DMU_o exhibits NIRS iff A and B exhibit CRS and DRS, respectively.

Con. 6. DMU_o exhibits VRS iff A and B exhibit IRS and DRS, respectively.

We draw the attention to the fact that the following conditions *never* occurs for DMU_o:

Con. 7. A and B exhibit DRS and IRS, respectively.

Con. 8. A and B exhibit DRS and CRS, respectively.

Con. 9. A and B exhibit CRS and IRS, respectively.

For example, DMU_A exhibits IRS because of Con. 1. The last column of Table 1 reports the RTS estimation of all the DMUs in terms of the above conditions.

As illustrated in Section 3, NDRS, NIRS and VRS individually encompass different RTS (CRS, IRS or DRS). Therefore, when DMU_o according to Con. 4, 5, or 6 is classified as having either NDRS, NIRS or VRS, we present three propositions regarding sensitivity analysis to identify its different partitions.

Proposition 1. If DMU_o exhibits NDRS (involving CRS and IRS) derived from Con. 4, then, its CRS and IRS classifications are unaltered under the following conditions, respectively:

(I) IRS prevails when the inputs and outputs of DMU_o lie within $[x_{io}^L, x_{io}^U]$ and $[y_{ro}^L, y'_{ro})$ where $y'_{ro} = \min\{\sigma_o^{A*} y_{ro}^L, y_{ro}^U\}$ and σ_o^{A*} is the optimal solution of (10).

(II) CRS prevails when the inputs and outputs of DMU_o lie within $[x_{io}^L, x_{io}^U]$ and $[y'_{ro}, y_{ro}^U]$ where $y'_{ro} = \min\{\sigma_o^{A*} y_{ro}^L, y_{ro}^U\}$ and σ_o^{A*} is the optimal solution of (10).

Proof (I). Assume that $\text{DMU}'_o = (x_{io}^L, \alpha y_{ro}^L)$ where $\alpha \in [1, \sigma_o^{A*})$ and its RTS is not IRS. Therefore, the RTS on DMU'_o will be either DRS or CRS. In this regard, $\text{DMU}''_o = (\alpha \theta_o^{*A} x_{io}^L, \alpha y_{ro}^L)$ will be either DRS or CRS because the inputs change does not impact on RTS. Therefore, we have:

$$
\begin{aligned}
&\sum_{j \in E_o} \lambda_j^* x_{ij}^L \le \varphi^* \left(\alpha \theta_o^{*A} x_{io}^L\right) \le \alpha \theta_o^{*A} x_{io}^L, \qquad i = 1, \ldots, m, \\
&\sum_{j \in E_o} \lambda_j^* y_{rj}^U \ge \left(\alpha y_{ro}^L\right), \qquad r = 1, \ldots, s, \\
&\sum_{j \in E_o} \lambda_j^* \ge 1, \qquad\qquad (i) \\
&\lambda_j^* \ge 0, \qquad j \in E_o.
\end{aligned}
$$

where φ^* is the optimal objective function value of model (8) when assessing DMU'_o. Evidently, $\left(\lambda_j^*/\alpha\right) \geq 0 \ \left(j \in E_o\right)$ is the feasible solution for model (10). According to the above equation (i), $\left(\sum_{j\in E_o} \lambda_j^*/\alpha\right) \geq 1/\alpha > 1/\sigma_o^{A*}$ failing the optimality of (10). ■

Proof (II). We need to prove that $(x_{io}^L, \alpha y_{ro}^U), \left(\sigma_O^{A*} y_{ro}^L/y_{ro}^U \leq \alpha \leq 1\right)$ exhibits CRS where (x_{io}^L, y_{ro}^U) is CRS. Obviously, the RTS of $(x_{io}^L, \alpha y_{ro}^U)$ and $(\alpha x_{io}^L, \alpha y_{ro}^U)$ are identical. In addition, $\left(\left(\sigma_O^{A*} y_{ro}^L/y_{ro}^U\right) x_{io}^L, \left(\sigma_O^{A*} y_{ro}^L/y_{ro}^U\right) y_{ro}^U\right)$ exhibits CRS since (x_{io}^L, y_{ro}^U) is CRS (see Thrall and Banker, 1992). As a result, $(x_{io}^L, \alpha y_{ro}^U), \left(\sigma_O^{A*} y_{ro}^L/y_{ro}^U \leq \alpha \leq 1\right)$ exhibits CRS. ■

Proposition 2. If DMU_o exhibits NIRS (involving CRS and DRS) derived from Con. 5, then its CRS and DRS classifications are unaltered under the following conditions, respectively:

(I) DRS prevails when the inputs and outputs of DMU_o lie within $[x_{io}^L, x_{io}^U]$ and $(y''_{ro}, y_{ro}^U]$ where $y''_{ro} = \max\{\tau_o^{B*} y_{ro}^U, y_{ro}^L\}$ and τ_o^{B*} is the optimal solution of (11).

(II) CRS prevails when the inputs and outputs of DMU_o lie within $[x_{io}^L, x_{io}^U]$ and $[y_{ro}^L, y''_{ro}]$ where $y''_{ro} = \max\{\tau_o^{B*} y_{ro}^U, y_{ro}^L\}$ and τ_o^{B*} is the optimal solution of (11).

Proof (I). Assume that $\text{DMU}'_o = (x_{io}^L, \alpha y_{ro}^L)$ where $\alpha \in (\tau_o^{B*}, 1]$ and its RTS is not DRS. Therefore, the RTS on DMU'_o will be either IRS or CRS. In this regard, $\text{DMU}''_o = (\alpha\theta_o^{*B} x_{io}^L, \alpha y_{ro}^U)$ will be either IRS or CRS because the inputs change does not impact on RTS. Therefore, we have:

$$\sum_{j\in E_o} \lambda_j^* x_{ij}^L \leq \varphi^*\left(\alpha\theta_o^{*B} x_{io}^L\right) \leq \alpha\theta_o^{*B} x_{io}^L, \qquad i = 1,\ldots,m,$$

$$\sum_{j\in E_o} \lambda_j^* y_{rj}^U \geq \alpha y_{ro}^U, \qquad r = 1,\ldots,s,$$

$$\sum_{j\in E_o} \lambda_j^* \leq 1, \qquad (i)$$

$$\lambda_j^* \geq 0, \qquad j \in E_o.$$

where φ^* is the optimal objective function value of model (9) when assessing DMU_o'. Evidently, $\left(\lambda_j^*/\alpha\right)\geq 0\ \left(j\in E_o\right)$ is the feasible solution for model (11). According to the above equation (i), $\left(\sum_{j\in E_o}\lambda_j^*/\alpha\right)\leq 1/\alpha < 1/\tau_o^{\mathrm{B}*}$ failing the optimality of (11). ■

Proof (II). We need to prove that $(x_{io}^L,\alpha y_{ro}^L),\left(1\leq\alpha\leq\tau_O^{\mathrm{B}*}y_{ro}^U/y_{ro}^L\right)$ exhibits CRS where (x_{io}^L,y_{ro}^L) is CRS. Obviously, the RTS of $(x_{io}^L,\alpha y_{ro}^L)$ and $(\alpha x_{io}^L,\alpha y_{ro}^L)$ are identical. In addition, $\left(\left(\tau_O^{\mathrm{B}*}y_{ro}^U/y_{ro}^L\right)x_{io}^L,\left(\tau_O^{\mathrm{B}*}y_{ro}^U/y_{ro}^L\right)y_{ro}^L\right)$ exhibits CRS since (x_{io}^L,y_{ro}^L) is CRS (see Thrall and Banker, 1992). As a result, $(x_{io}^L,\alpha y_{ro}^L),\left(1\leq\alpha\leq\tau_O^{\mathrm{B}*}y_{ro}^U/y_{ro}^L\right)$ exhibits CRS. ■

Proposition 3. If DMU_o exhibits VRS (involving CRS, IRS and DRS) derived from Con. 6, its CRS, IRS and DRS classifications are unaltered under the following conditions, respectively:

(I) DRS prevails when the inputs and outputs of DMU_o lie within $[x_{io}^L,x_{io}^U]$ and $(\tau_o^{\mathrm{B}*}y_{ro}^U,y_{ro}^U]$ and $\tau_o^{\mathrm{B}*}$ is the optimal solution of (11).

(II) CRS prevails when the inputs and outputs of DMU_o lie within $[x_{io}^L,x_{io}^U]$ and $[\sigma_o^{A*}y_{ro}^L,\tau_o^{\mathrm{B}*}y_{ro}^U]$ and σ_o^{A*} is the optimal solution of (10).

(III) IRS prevails when the inputs and outputs of DMU_o lie within $[x_{io}^L,x_{io}^U]$ and $[y_{ro}^L,\sigma_o^{A*}y_{ro}^L]$ and σ_o^{A*} is the optimal solution of (10).

Proof (I). The proof is straightforward according to Proposition 2 (omitted). ■
Proof (II). The proof is straightforward according to Propositions 1 and 2 (omitted). ■
Proof (III). The proof is straightforward according to Proposition 1 (omitted). ■

Let us show the detailed formulation and solution for studying three DMUs, $\mathrm{DMU_B}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_F}$, in the example of section 3 (see Fig. 1) that exhibit NDRS, NIRS and VRS and explains how to implement a sensitivity analysis.

The set of efficient DMUs is $E_o=\{\mathrm{DMU_B},\ \mathrm{DMU_C}\}$ since their upper efficiency scores derived from model (9) are equal to unity (see the eighth column of Table 2). From Table 2, $\mathrm{DMU_B}$,

DMU_E and DMU_F display NDRS, NIRS and VRS, respectively. The problem (10) for DMU_B with the situation for NDRS is formulated as follows:

$$(\sigma_B^A)^{-1} = \max \sum_{j=1}^{2} \hat{\lambda}_j$$
$$\begin{aligned} s.t. \quad & 3\lambda_1 + 6\lambda_2 \leq 0.33 \times 3 \\ & 3\lambda_1 + 6\lambda_2 \geq 1 \\ & \lambda_1, \lambda_2 \geq 0. \end{aligned}$$

where A=(3,1), $\theta_B^{A*} = 0.33$ and the optimal value of the above model, $(\sigma_B^A)^{-1}$, for DMU_B is 0.167. According to proposition 1, the region of IRS and CRS for DMU_B can be obtained as:

(I) IRS prevails at DMU_B when its input and output vary within [3, 6] and [1, 3), respectively. Note that $y'_{rB} = \min\{5.988, 3\} = 3$.

(II) CRS prevails at DMU_B when its inputs of DMU_o vary within [3, 6] and its output is 3. Note that $[y'_{rB}, y^U_{rB}] = [3, 3]$ where $y'_{rB} = \min\{5.988, 3\} = 3$.

Model (11) for DMU_E with the situation for NIRS is described as follows:

$$(\tau_E^B)^{-1} = \min \sum_{j=1}^{2} \hat{\lambda}_j$$
$$\begin{aligned} s.t. \quad & 3\lambda_1 + 6\lambda_2 \leq 0.65 \times 10 \\ & 3\lambda_1 + 6\lambda_2 \geq 6.5 \\ & \lambda_1, \lambda_2 \geq 0. \end{aligned}$$

where B=(10, 6.5), $\theta_E^{B*} = 0.65$ and the optimal value of the above model, $(\tau_E^B)^{-1}$, for DMU_E is 1.0833. According to Proposition 2, the region of DRS and CRS for DMU_B can be determined as:

(I) DRS prevails at DMU_E when its input and output vary within [10,12] and (6,6.5], respectively. Note that $y''_{rE} = \max\{6, 6\} = 6$.

(II) CRS prevails at DMU_E when its input and output vary within [10,12] and [5,6], respectively. Note that $y''_{rE} = \max\{6, 6\} = 6$.

According to the proposed algorithm, we use models (10) and (11) for DMU_F with the situation for VRS as follows:

$$
\begin{aligned}
(\sigma_F^A)^{-1} = \max & \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t.\quad & 3\lambda_2 + 6\lambda_3 \le 0.142 \times 7 \\
& 3\lambda_2 + 6\lambda_3 \ge 1 \\
& \lambda_1, \lambda_2 \ge 0,
\end{aligned}
\qquad\qquad
\begin{aligned}
(\tau_F^B)^{-1} = \min & \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t.\quad & 3\lambda_2 + 6\lambda_3 \le 0.92 \times 7 \\
& 3\lambda_2 + 6\lambda_3 \ge 6.5 \\
& \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

where A=(7,1), B=(7, 6.5), $\theta_F^{A*} = 0.142$, $\theta_E^{B*} = 0.92$ and the optimal values of the above models are $(\sigma_F^A)^{-1} = 0.3333$ and $(\tau_F^B)^{-1} = 1.0833$. According to proposition 3, the region of CRS, IRS and DRS for DMU_B can be determined as:

(I) DRS prevails at DMU_F when the inputs and outputs of DMU_o lie within [7,10] and (6,6.5], respectively.

(II) CRS prevails at DMU_F when the inputs and outputs of DMU_o lie within [7,10] and [3,6], respectively.

(III) IRS prevails at DMU_F when the inputs and outputs of DMU_o lie within [7,10] and [1, 3), respectively.

5. Stability of the RTS classification with interval data

This section presents the stability regions of RTS classification when inputs and outputs are given by intervals. Given the identification of the RTS of DMUs in the input-oriented perspective, the input perturbations cannot change the RTS classification and we only need to study the output perturbations. Let us graphically explain the sensitivity analysis of the RTS classification.

First we consider DMU_E in Fig. 1, DMU_E is classified as NIRS (i.e., its points B and A are DRS and CRS, respectively). The output increase of DMU_E expresses the output increase[1] of point B in DMU_E and this DMU still exhibits DRS. Therefore, the RTS of DMU_E remains unchanged since the RTS of point B is unchanged with the output augmentation. The output reduction in DMU_E prompts the output reduction[2] of point A in DMU_E and this change is able to turn RTS into VRS, but it definitely depends on the amount of the output reduction. As a result, the impact of the output perturbation on two points A and B of the DMU under evaluation enable us to find

[1] The output increase means an increase in the upper bound of the output.
[2] The output reduction means a reduction in the lower bound of the output.

the stability of RTS classifications. We here explain how to identify the stability region of six types of RTS:

(1) *A DMU with IRS*: If a DMU exhibits IRS then the reduction of its outputs does not alter the RTS classification and only the output augmentation has the capability to change the RTS classification. We identify the stability region for the virtual point B using model (11) and Theorem 4 when its outputs are increased. In this regard, the IRS of the point B remains unaltered for $1 \le \alpha \le \sigma_o^{B*}$ where α is a proportional increase in all outputs of the virtual point B ($y_{rB}^{new} = \alpha y_{rB}$). Therefore, the stability region of IRS for the DMU is $[y_{ro}^L, \alpha y_{ro}^U]$ where $1 \le \alpha < \sigma_o^{B*}$.

(2) *A DMU with DRS*: If a DMU exhibits DRS then the augmentation of its outputs does not alter the RTS classification and only the output reduction is able to change the RTS classification. We identify the stability region for the virtual point A using model (10) and Theorem 5 when its outputs are reduced. In this regard, the DRS of the point A remains unaltered for $\tau_o^{A*} \le \beta \le 1$ where β is a proportional reduction in all outputs of the virtual point A ($y_{rA}^{new} = \beta y_{rA}$). Therefore, the stability region of DRS for the DMU is $[\beta y_{ro}^L, y_{ro}^U]$ where $\tau_o^{A*} < \beta \le 1$.

(3) *A DMU with CRS*: If a DMU exhibits CRS then the variation of its outputs (increase and/or decrease) may alter the RTS classification. We identify the stability region for the virtual points A and B using models (10) and (11) as well as Theorem 3 when its outputs are varied. The stability region of CRS for the DMU is $[\beta y_{ro}^L, \alpha y_{ro}^U]$ where $(\min\{1, \sigma_o^{A*}\} \le \beta \le 1)$ and $(1 \le \alpha \le \max\{1, \tau_o^{B*}\})$.

(4) *A DMU with NDRS*: If a DMU exhibits NDRS then the reduction of its outputs does not alter the RTS classification and only the output augmentation has the capability to change the RTS classification. We identify the stability region for the virtual point B using model (11) and Theorem 3 when its outputs are increased. The stability region of NDRS for the DMU is $[y_{ro}^L, \alpha y_{ro}^U]$ where $(1 \le \alpha < \max\{1, \tau_o^{B*}\})$.

(5) *A DMU with NIRS*: If a DMU exhibits NIRS then the augmentation of its outputs does not alter the RTS classification and only the output reduction is able to change the RTS classification. We identify the stability region for the virtual point A using model (10) and

Theorem 3 when its outputs are reduced. The stability region of NIRS for the DMU is $[\beta y_{ro}^{L}, y_{ro}^{U}]$ where $(\min\{1, \sigma_o^{A*}\} < \beta \leq 1)$.

(6) *A DMU with VRS*: If a DMU is VRS the variation of its outputs (increase and/or decrease) does not alter the RTS classification.

Let us consider all six DMUs of an earlier example in Section 3 (see Fig. 1) to detail the formulation and solution to the sensitivity analysis of RTS classification.

The set of efficient DMUs is E_o={DMU_B, DMU_C} because their upper efficiency scores flowed from model (7) are equal to 1.

(1) DMU_A shows IRS. We evaluate the point B=(2.5,2) using model (11) as follows:

$$
\begin{aligned}
(\sigma_A^B)^{-1} = \max & \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \leq 0.8 \times 2.5 \\
& 3\lambda_2 + 6\lambda_3 \geq 2 \\
& \lambda_1, \lambda_2 \geq 0.
\end{aligned}
$$

The optimal solution of the above model is $(\sigma_A^{B*})^{-1} = 0.666$ and the RTS keeps unchanged when the output of DMU_A varies within [1, 3).

(2) DMU_B exhibits NDRS. We evaluate the point B=(3,3) using model (11) as follows:

$$
\begin{aligned}
(\sigma_B^B)^{-1} = \max & \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \leq 3 \\
& 3\lambda_2 + 6\lambda_3 \geq 3 \\
& \lambda_1, \lambda_2 \geq 0.
\end{aligned}
$$

The optimal solution of the above model is $(\sigma_B^{B*})^{-1} = 1$ and the RTS keeps unchanged when the output of DMU_B varies within [1, 3].

(3) DMU_C exhibits CRS. We assess the points B=(6,6) and A=(6,4) using models (11) and (10), respectively, as follows:

$$(\tau_C^B)^{-1} = \min \sum_{j=1}^{2} \lambda_j \qquad\qquad (\sigma_C^A)^{-1} = \max \sum_{j=1}^{2} \lambda_j$$

$$\begin{aligned} s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 6 \\ & 3\lambda_2 + 6\lambda_3 \ge 6 \\ & \lambda_1, \lambda_2 \ge 0. \end{aligned} \qquad\qquad \begin{aligned} s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 0.33 \times 6 \\ & 3\lambda_2 + 6\lambda_3 \ge 4 \\ & \lambda_1, \lambda_2 \ge 0. \end{aligned}$$

The optimal solution of the above models are $(\tau_C^{B*})^{-1} = 1$ and $(\sigma_C^{A*})^{-1} = 1.3333$. Therefore, the RTS keeps unchanged when the output of DMU_C varies within [3, 6].

(4) DMU_D exhibits DRS. We evaluate the point A=(7.5,6.5) using model (10) as follows:

$$\begin{aligned} & (\tau_D^A)^{-1} = \min \sum_{j=1}^{2} \lambda_j \\ s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 0.86 \times 7.5 \\ & 3\lambda_2 + 6\lambda_3 \ge 6.5 \\ & \lambda_1, \lambda_2 \ge 0, \end{aligned}$$

The optimal solution of the above model is $(\tau_D^{A*})^{-1} = 1.0833$. Therefore, the RTS keeps unchanged when the output of DMU_D varies within (6, 7].

(5) DMU_E exhibits NIRS. We evaluate the point A=(10,5) using model (10) as follows:

$$\begin{aligned} & (\sigma_E^A)^{-1} = \max \sum_{j=1}^{2} \lambda_j \\ s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 0.5 \times 10 \\ & 3\lambda_2 + 6\lambda_3 \ge 5 \\ & \lambda_1, \lambda_2 \ge 0, \end{aligned}$$

The optimal solution of the above model is $(\sigma_E^{A*})^{-1} = 1.6666$. Therefore, the RTS keeps unchanged when the output of DMU_E varies within [3,6.5].

(6) DMU_F exhibits VRS and its RTS preserves fixed with any output variations.

6. Illustrative Example

In order to illustrate our method as well as to compare with the conventional approach, this section presents a real-world data set with 28 Chinese cities (DMUs) in 1983 from Charnes et al. (1989). Seiford and Zhu (1999b) used the data set to implement their RTS sensitivity analysis on the inefficient DMUs without perceiving imprecision in data as presented in the 12th column

Table 3. We think of impression as a variable having a "true" value added a percentage of uncertainty, +/- 5%, in data of 28 DMUs in 1983, as represented in Table 2.

-------**Insert Table 2 here**-------

We first calculate the efficiency of points A and B for each DMU using models (8) and (9) as reported in the 2nd and 3rd columns in Table 3. The DMUs {DMU_1, DMU_{11}, DMU_{19}, DMU_{21}, DMU_{22}, DMU_{23}, DMU_{24}, DMU_{25}, DMU_{26}, DMU_{28}} are efficient on account of $q_j^{B*} = 1$. The 4th column of Table 3 presents the set of DMUs which their λs are strictly positive in model (4). To determine the RTS of DMU we solve models (10) and (11) with respect to the efficiency of points A and B as its the optimal objective function value are reported in 5th, 6th, 8th and 9th columns of Table 3. The 7th, 10th and 11th columns of Table 3 represent the RTS of points A and B while the 11th column indicates the RTS of DMU under assessment in terms of $(s_o^{A*})^{-1}$, $(t_o^{B*})^{-1}, (s_o^{A*})^{-1}$ and conditions1-6.

-------**Insert Table 3 here**-------

As a result, DMUs {DMU_1, DMU_{11}, DMU_{19}, DMU_{21}, DMU_{22}, DMU_{23}, DMU_{24}, DMU_{25}, DMU_{26}, DMU_{28}} exhibit NDRS where all of them are a composite of two partitions IRS and CRS, DMUs {DMU_6} exhibit VRS where all of them are a mix of three partitions: CRS, IRS and DRS, DMUs {DMU_2, DMU_3, DMU_4, DMU_5} exhibit DRS, and the remaining DMUs exhibit IRS. Table 4 shows the different RTS regions of DMUs whose RTS exhibit NDRS and VRS in terms of their outputs. Notice that the change in inputs does not influence the RTS classification.

-------**Insert Table 4 here**-------

The stability region of DMUs with the interval observations keeping their present RTS classifications are identified by means of the discussion in section 5. The results are presented in the last column on Table 3.

Even though we have introduced imprecision in the data set (+/- 5%) the overall structure of the RTS classification remains the same as under the conventional approach with precise data (comparing columns 11 and 12 of Table 3). Adding data imprecision we obtain a more nuanced picture where, for instance, DMU 1 is now classified as NDRS whereas with precise data it was classified as CRS and DMU 6 is now VRS versus CRS with precise data etc. Further nuances are illustrated by the stability regions (column 13 in Table 3) and mapping of RTS regions in Table 4.

It is important to emphasize that our approach does not generate useless vague results where most DMUs are classified as exhibiting every kind of RTS despite adding imprecision to the data set. Clearly this is caused by the fact that we still refer to a single common efficient frontier.

7. Final remarks

In the present paper we have shown how one could can extend the RTS classification of standard DEA models to Imprecise DEA where the input and output data take the form of intervals. In short, our idea relates RTS classification to one common frontier for all DMUs and that frontier becomes the frontier spanned by the most optimistic data for all DMUs. Once this frontier is in place we can utilize RTS characterizations of standard (crisp) DEA models and analyze the sensitivity of these using the Seiford-Zhu approach (Seiford and Zhu, 1999b). We illustrate our approach on a well know data set from Charnes et al. (1989).

Fundamentally, our approach builds on the assumption that the data imprecision represented by the interval data can be seen as a relative uncertainty around some "true" value of the variables. However, many other forms of imprecision can be imagined and we leave for future research a deeper analysis of the connection between imprecision measures and RTS classification.

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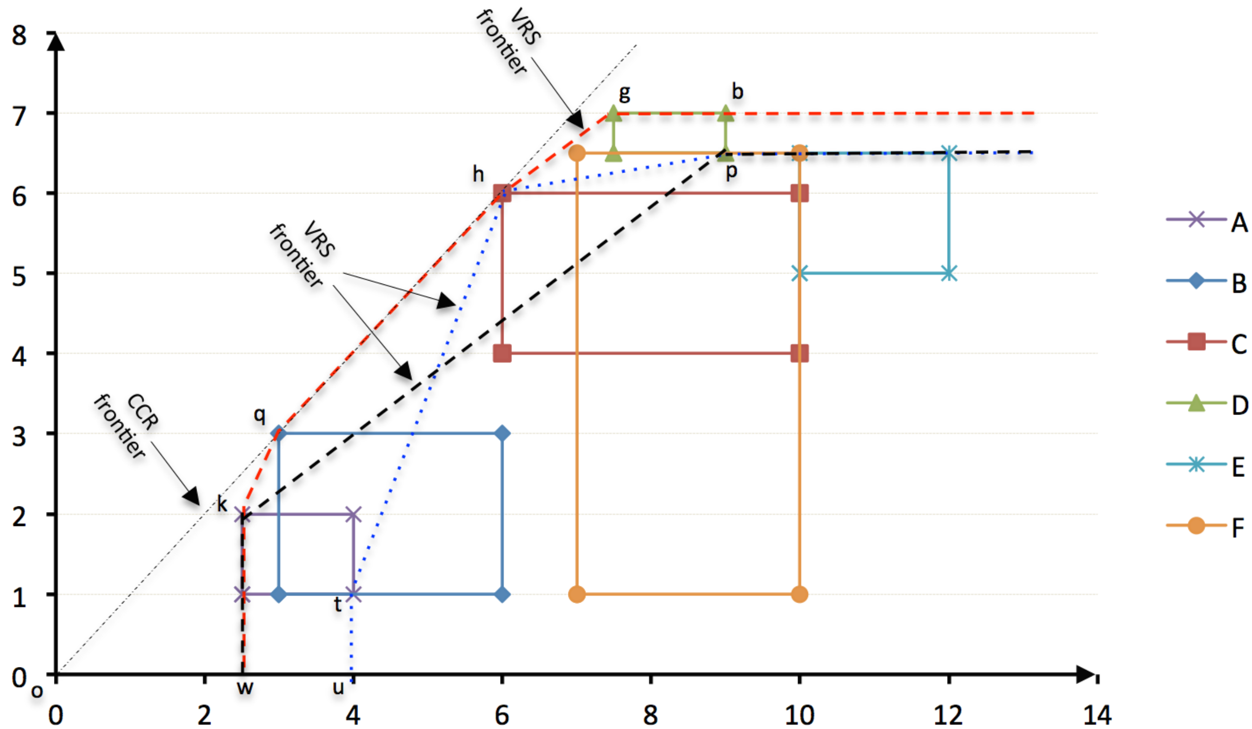

Fig 1. Different production possibility frontiers for six DMUs

Table 1. Efficiency and RTS classification of points A and B, and RTS classification of DMUs

DMU	A	q^{*A} model (7)	$(s_o^{A*})^{-1}$	$(t_o^{A*})^{-1}$	RTS of A	B	q^{*B} model (6)	$(s_o^{B*})^{-1}$	$(t_o^{B*})^{-1}$	RTS of B	RTS
A	[2.5,1]	0.400	0.333	0.166	IRS	[2.5,2]	0.800	0.666	0.333	IRS	IRS
B	[3,1]	0.333	0.333	0.166	IRS	[3,3]	1.000	1.000	0.500	CRS	NDRS
C	[6,4]	0.666	1.333	0.666	CRS	[6,6]	1.000	2.000	1.000	CRS	CRS
D	[7.5,6.5]	0.866	2.166	1.083	DRS	[7.5,7]	0.933	2.333	1.166	DRS	DRS
E	[10,5]	0.500	1.666	0.833	CRS	[11,6.5]	0.650	2.166	1.083	DRS	NIRS
F	[7,1]	0.142	0.333	0.166	IRS	[7,6.5]	0.928	2.166	1.083	DRS	VRS

Table 2. Imprecise data for 28 Chinese cities

DMU	Lower bound of inputs			Upper bound of inputs			Lower bound of outputs			Upper bound of outputs		
	$I^L(1)$	$I^L(2)$	$I^L(3)$	$I^U(1)$	$I^U(2)$	$I^U(3)$	$O^L(1)$	$O^L(2)$	$O^L(3)$	$O^U(1)$	$O^U(2)$	$O^U(3)$
1	463.068	1514338	683005.4	511.812	1673742	754900.7	7071515	1607495	1257610	7815885	1776705	1389990
2	356.6965	907367.8	495930.4	394.2435	1002880	548133.6	2676340	560120	965770	2958060	619080	1067430
3	260.0055	743455.8	352748.3	287.3745	821714.3	389879.7	2389155	421420	537890	2640645	465780	594510
4	198.379	465192.2	131116.2	219.261	514159.8	144917.9	1270245	154565	389785	1403955	170835	430815
5	189.9905	493701.7	135553.6	209.9895	545670.3	149822.4	1308720	221255	363755	1446480	244545	402045
6	172.805	456372.4	246137.4	190.995	504411.6	272046.6	1266920	199215	583775	1400280	220185	645225
7	143.3835	390190.7	90986.25	158.4765	431263.4	100563.8	720999.7	97748.35	283634.9	796894.4	108037.7	313491.2
8	178.8185	446598.8	127329.5	197.6415	493609.2	140732.6	1099693	158629.1	392341.5	1215451	175326.9	433640.6
9	120.156	281707.3	123808.8	132.804	311360.7	136841.3	925253.5	157753.2	280180.7	1022649	174358.8	309673.4
10	116.565	318562.6	100200.3	128.835	352095.5	110747.7	634701.7	79548.25	236246	701512.4	87921.75	261114
11	126.4735	318824.8	98259.45	139.7865	352385.3	108602.6	792870	122032.3	814024.6	876330	134877.8	899711.4
12	103.8445	263692.5	62610.7	114.7755	291449.6	69201.3	513406.6	81973.6	283272.9	567449.4	90602.4	313091.1
13	89.7275	194748.1	123831.6	99.1725	215247.9	136866.5	514826.9	82931.2	138110.1	569019.2	91660.8	152648
14	107.502	294290.1	109612	118.818	325268	121150.1	872582.6	160985.1	275993.1	964433.4	177930.9	305045
15	83.163	198737.2	61657.85	91.917	219656.9	68148.15	807458.2	122231.8	261232.9	892453.8	135098.3	288731.1
16	70.015	238030.1	81742.75	77.385	263085.9	90347.25	513814.2	124106.1	134623.6	567899.9	137169.9	148794.5
17	73.758	172013.7	50215.1	81.522	190120.4	55500.9	518761.8	77955.1	182470.3	573368.3	86160.9	201677.7
18	70.4805	150907.5	50079.25	77.8995	166792.5	55350.75	519765.9	133115.9	114702.1	574478.1	147128.1	126776
19	85.2055	193534	72751	94.1745	213906	80409	652063.9	191232.2	145237	720702.2	211361.9	160525.1
20	71.744	175442.2	74751.7	79.296	193909.8	82620.3	428205.9	84579.45	162566.9	473280.2	93482.55	179679.2
21	67.1935	129459.4	12752.8	74.2665	143086.7	14095.2	876443.4	58115.3	242900.8	968700.6	64232.7	268469.3
22	64.695	698517.9	11746.75	71.505	772046.1	12983.25	958299.2	130252.6	283781.2	1059173	143963.4	313652.9
23	55.651	90926.4	7081.3	61.509	100497.6	7826.7	631212.3	59479.5	206676.3	697655.7	65740.5	228431.7
24	65.8065	165994.5	13210.7	72.7335	183467.6	14601.3	1039188	92964.15	203374.1	1148576	102749.9	224781.9
25	45.5715	105994.4	9976.9	50.3685	117151.7	11027.1	673814.1	65875.85	142634.9	744741.9	72810.15	157649.1
26	64.3815	99821.25	9801.15	71.1585	110328.8	10832.85	659580.3	36103.8	243238	729009.8	39904.2	268842
27	19.0665	52614.8	1754.65	21.0735	58153.2	1939.35	154331.3	12198.95	27588.95	170576.7	13483.05	30493.05
28	68.7515	127134.7	4105.9	75.9885	140517.3	4538.1	341958.2	36925.55	191705.3	377953.8	40812.45	211884.8

Table 3. RTS classification and stability region

DMU	q_o^{A*}	q_o^{B*}	E_o	$(s_o^A)^{-1}$	$(t_o^A)^{-1}$	RTS A	$(s_o^B)^{-1}$	$(t_o^B)^{-1}$	RTS B	RTS of DMU	RTS (Charnes et al., 1989)	Stability region
1	0.905	1	1	1.105	1.105	IRS	1	1	CRS	NDRS	CRS	N.G.*
2	0.657	0.727	1,11,23	0.478	0.478	DRS	0.418	0.476	DRS	DRS	DRS	0.478<β≤1
3	0.583	0.645	1,11,23,24	0.887	0.887	DRs	0.772	0.818	DRS	DRS	DRS	0.887<β≤1
4	0.462	0.511	1,11,23,24	0.850	0.850	DRS	0.767	0.770	DRS	DRS	DRS	0.850<β≤1
5	0.505	0.558	1,11,22,23	0.952	0.952	DRS	0.802	0.875	DRS	DRS	DRS	0.952<β≤1
6	0.637	0.705	1,11,23,24	1.067	1.067	IRS	0.947	0.973	DRS	VRS	CRS	N.G.
7	0.399	0.441	1,11,22,24,25	1.434	1.434	IRS	1.292	1.340	IRS	IRS	IRS	1≤α≤1.292
8	0.475	0.525	1,11,23,24	1.125	1.125	IRS	1.121	1.121	IRS	IRS	CRS	1≤α<1.121
9	0.595	0.657	1,11,23	1.274	1.274	IRS	1.124	1.228	IRS	IRS	IRS	1≤α<1.124
10	0.415	0.459	1,11,22,24	1.933	1.933	IRS	1.641	1.819	IRS	IRS	IRS	1≤α<1.640
11	0.905	1	11	1.105	1.105	IRS	1	1	CRS	NDRS	IRS	N.G.
12	0.515	0.569	1,11,22,23	1.662	1.662	IRS	1.441	1.516	IRS	IRS	IRS	1≤α<1.441
13	0.437	0.483	1,23	2.247	2.247	IRS	1.987	2.087	IRS	IRS	IRS	1≤α<1.987
14	0.598	0.661	1,11,23,24	2.133	2.133	IRS	1.882	1.953	IRS	IRS	IRS	1≤α<1.882
15	0.733	0.810	1,11,23,24	1.500	1.500	IRS	1.353	1.359	IRS	IRS	IRS	1≤α<1.353
16	0.552	0.610	1,11,22,23	4.671	4.671	IRS	4.223	4.227	IRS	IRS	IRS	1≤α<4.223
17	0.555	0.613	1,11,23,24	2.190	2.190	IRS	1.974	1.985	IRS	IRS	IRS	1≤α<1.974
18	0.850	0.940	19,23	1.293	1.293	IRS	1.168	1.188	IRS	IRS	IRS	1≤α<1.168
19	0.905	1	19	1.105	192.926	IRS	1	1	CRS	NDRS	IRS	N.G.
20	0.535	0.591	1,11,23	2.647	2.647	IRS	2.393	2.401	IRS	IRS	IRS	1≤α<2.393
21	0.905	1	21	1.105	1.105	IRS	1	1	CRS	NDRS	CRS	1≤α≤1
22	0.905	1	22	1.105	1.105	IRS	1	1	CRS	NDRS	IRS	N.G.
23	0.905	1	23	1.105	1.105	IRS	1	1	CRS	NDRS	CRS	N.G.
24	0.905	1	24	1.105	1.105	IRS	1	1	CRS	NDRS	CRS	N.G.
25	0.905	1	25	1.105	1.105	IRS	1	1	CRS	NDRS	CRS	N.G.
26	0.905	1	26	1.105	1.105	IRS	1	1	CRS	NDRS	CRS	N.G.
27	0.893	0.987	23	4.520	4.520	IRS	2.323	4.140	IRS	IRS	IRS	1≤α<2.323
28	0.905	1	28	1.105	1.105	IRS	1	1	CRS	NDRS	IRS	N.G.

* Non-Change

Table 4. RTS regions for NDRS and VRS

DMU	RTS	IRS			CRS			DRS		
		O(1)	O(2)	O(3)	O(1)	O(2)	O(3)	O(1)	O(2)	O(3)
1	NDRS	[7071515, 7815885)	[1607495, 1776705)	[1257610, 1389990)	[7815885, 7815885]	[1776705, 1776705]	[1389990, 1389990]	-	-	-
6	VRS	[1266920, 1351292)	[199215, 212482)	[583775, 622652.3)	[1351292, 1363114]	[212482, 214340.9]	[622652.3, 628099.592]	(1363114, 1400280]	(214340.9, 220185]	(628099.592, 645225]
11	NDRS	[792870, 876330)	[122032.3, 134877.8)	[814024.6, 899711.4)	[792870, 876330]	[122032.3, 134877.8]	[814024.6, 899711.4]	-	-	-
19	NDRS	[652063.9, 720702.2)	[191232.2, 211361.9)	[145237, 160525.1)	[720702.2, 720702.2]	[211361.9, 211361.9]	[160525.1, 160525.1]	-	-	-
21	NDRS	[876443.4, 968700.6)	[58115.3, 64232.7)	[242900.8, 268469.3)	[968700.6, 968700.6]	[64232.7, 64232.7]	[968700.6, 968700.6]	-	-	-
22	NDRS	[958299.2, 1059173)	[130252.6, 143963.4)	[283781.2, 313652.9)	[1059173, 1059173)	[143963.4, 143963.4)	[313652.9, 313652.9]	-	-	-
23	NDRS	[631212.3, 697655.7)	[59479.5, 65740.5)	[206676.3, 228431.7)	[697655.7, 697655.7]	[65740.5, 65740.5]	[228431.7, 228431.7]	-	-	-
24	NDRS	[1039188, 1148245)	[92964.15, 102720.2)	[203374.1, 224717)	[1148245, 1148576]	[102720.2, 102749.9]	[224717, 224781.9]	-	-	-
25	NDRS	[673814.1, 744741.9)	[65875.85, 72810.15)	[142634.9, 157649.1)	[744741.9, 744741.9]	[72810.15, 72810.15]	[157649.1, 157649.1]	-	-	-
26	NDRS	[659580.3, 728934.4)	[36103.8, 39900.07)	[243238, 268814.2)	[728934.4, 729009.8]	[39900.07, 39904.2]	[268814.2, 268842]	-	-	-
28	NDRS	[341958.2, 377953.8)	[36925.55, 40812.45)	[191705.3, 211884.8)	[377953.8, 377953.8]	[40812.45, 40812.45]	[211884.8, 211884.8]	-	-	-