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# Ranking Production Units According to Marginal Efficiency Contribution

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**Abstract**

League tables associated with various forms of service activities from schools to hospitals illustrate the public need for ranking institutions by their productive performance. We present a new method for ranking production units which is based on each units marginal contribution to the technical efficiency of various "mergers" relative to a common reference technology. The approach is radically different from the usual one based on super-efficiency indexes in DEA. We illustrate the mechanics of our method by a series of numerical examples and further demonstrate that our new index inherits all relevant and desirable properties of the Farrell efficiency index upon which it is constructed.

**Keywords:** Production units, Ranking, Marginal efficiency contribution, DEA, Super efficiency, Contribution index

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# 1 Introduction

Data Envelopment Analysis (DEA) has proved to be one of the most popular methods for assessing the productive performance of a given sample of production units, see e.g., Cooper et al., (2007) for a survey of models and applications. Productive performance is here tantamount to efficiency and the assessment is relative in the sense that each unit in the sample is assigned an efficiency score (between 0 and 1) indicating the factor by which either its input use or output production needs to be scaled in order to be as efficient as the best performing peers of the sample. As such, efficient units all get a DEA score of 1 while inefficient units all get DEA scores smaller than 1 (the smaller the more inefficient).

Many empirical studies are concerned with ranking the involved production units based on their associated efficiency scores (often broadly interpreted). In practice, league tables seem more popular than ever with on-line ranking results on anything from kindergartens and hospitals to schools and universities. The DEA literature offers a wide variety of such ranking methods as e.g., surveyed in Adler et al., (2002).

For inefficient units a complete ranking follows directly from their DEA scores, but we cannot rank the efficient units according to their scores simply because these are all truncated at value 1. Consequently, several studies have tried to develop indexes ranking these DEA efficient units. The seminal paper by Andersen and Petersen (1993) submit that efficient units can be ranked according to their influence on the spanning of the efficient frontier of the estimated production possibility set (which in DEA consists of the free disposal convex hull/cone of the data points). Specifically, they suggest to use the radial Farrell efficiency index (as used in standard DEA; see Farrell, 1957, Charnes et al., 1978) with respect to a given production unit relative to the technology estimated by excluding this unit from the sample. If the unit is efficient, excluding it from the data set makes it infeasible (super efficient) relative to the new (reduced) reference technology. Thus, the index was dubbed a *super efficiency index* and it results in index values weakly greater than one for efficient units while the index value is identical to the standard DEA score for all the inefficient units. In this way it is possible to obtain a complete ranking of all the units in the sample based on their efficiency/super-efficiency scores. Variations of this approach can, for instance, be found in the (different) slack-based approaches of Tone (2002) and Bogetoft and Hougaard (2004).

Yet, the different super efficiency indexes all have drawbacks in relation to obtaining a consistent ranking of the production units:

- Super efficiency indexes are not always well defined.[1]

- Using a super efficiency index the efficient units are in effect measured relative to different frontiers (i.e., the frontier of the technology spanned by the data set without the unit in question) while the inefficient units are all measured relative to the same frontier spanned by the entire data set. This inconsistency in the reference technology is making a direct comparison of super efficiency scores, and thereby its induced ranking, somewhat questionable.

- Super efficiency indexes measure the influence of given units on the spanning of the frontier, but it is well recognized that not all frontier units may be proper benchmarks.[2] In a ranking context this has the unfortunate consequence that frontier units are always ranked above non-frontier units despite that fact that a non-frontier unit may be very close to efficiency and optimal scale size while some frontier units are associated with extreme optimal weights (multipliers) and very far from optimal scale.

In the present paper we try to overcome these drawbacks by designing an index for ranking production units, which refers to the same underlying benchmark technology and is well defined for all units; efficient as well as inefficient. Moreover, it indicates the actual influence on the efficiency of the entire sample by measuring the marginal efficiency contribution of given units. In this way we obtain a complete and consistent ranking of the entire sample based on the units efficiency scores.

To be more precise, we assume that all mergers involving the production units in the sample are possible. The production of a merger is simply found by adding up the inputs and outputs of all the units involved in the merger. Each unit can only enter into a merged unit once. This is basically a very

[1]See e.g., the discussion in Dula and Hickman (1997), Lovell and Rouse, (2003).

[2]Due to the weak notion of dominance some units appear as frontier units although their supporting (dual) weights are unrealistic, see e.g., the discussion in Sexton et al., (1986).

weak technological assumption since, for instance, we do not assume integer constant returns to scale, where the same unit is replicated. The sample of all original production units as well as all the possible mergers is dubbed the *extended sample*, and a common reference technology, denoted $\widetilde{T}$, can be estimated as the free disposal convex hull of this extended sample of data points.

The idea is now to rank all the original production units according to their marginal contribution to the efficiency of various mergers. The marginal efficiency contribution of unit $j$ to a merged unit $S$ is computed as the difference between the efficiency score of mergers $S \cup j$ and $S$ relative to $\widetilde{T}$. That is, the marginal efficiency contribution of $j$ to $S$ reveals how the efficiency score of merger $S$ changes by adding unit $j$ to the merger.

In particular, we suggest to look at two indexes: The *marginal contributions index*, $I^M$, considers the marginal change in efficiency from merging production unit $j$ with the complement merger $N \setminus j$; The *average contributions index*, $I^A$, acknowledges that the marginal efficiency change of adding unit $j$ differs with respect to all the various mergers in the extended sample and suggests to look at the average of all these efficiency changes. In other words, the average contribution index of unit $j$ can be interpreted as the average efficiency change arising if $j$ was allowed to merge with any other coalition of units.

The latter index has some resemblance to the Shapley value of a transferable utility game (Shapley, 1953) where each production unit is a "player" and coalitional value is given by the Farrell efficiency score of the coalitions aggregate production relative to the extended technology.

We show that both suggested contributions indexes are well defined for all production units of the original sample and satisfy desirable properties such as scale invariance, continuity and monotonicity.

The rest of the paper is organized as follow; Section 2 presents the model and defines our common reference technology. Section 3 defines our two contributions indexes. Section 4 illustrates the approach by numerical examples. Section 5 looks at various desirable properties of the contribution indexes; and finally Section 6 closes with some remarks on computational complexity and possible extensions.

## 2 The Model

Let $x \in \mathbf{R}^s_+$ be a $s$-dimensional vector of inputs, let $y \in \mathbf{R}^t_+$ be a $t$-dimensional vector of outputs and let $z = (x, y) \in \mathbf{R}^s_+ \times \mathbf{R}^t_+$ be a feasible production plan, i.e., $x$ can produce $y$. Let $N = \{1, \ldots, n\}$ be a (non-empty) finite set of production units and let $Z = \{z^j\}_{j \in N}$ be a set of feasible production plans from these $n$ units.

For $j \in N$ let,

$$U(j) = \{i \in N \setminus \{j\} \mid x^i \leq x^j, y^i \geq y^j\} \tag{1}$$

be the set of units dominating production unit $j$, i.e., units that use weakly less inputs to produce weakly more outputs than $j$. Consequently, the set of undominated production units in the sample $Z$, is given by

$$N^E = \{j \in N \mid U(j) = \emptyset\} \subseteq N. \tag{2}$$

For a given merger of production units $S \subseteq N,\ S \neq \emptyset$, let

$$z(S) = (x(S), y(S)) = (\sum_{j \in S} x^j, \sum_{j \in S} y^j) \tag{3}$$

be the associated aggregate production plan of the merged unit $S$ and let

$$\widetilde{Z} = \{z(S)\}_{S \subseteq N} \tag{4}$$

be the extended sample including aggregate production plans for all possible mergers of $N$ (expect for $\emptyset$). There are $2^n - 1$ such possible mergers disregarding the empty set.

Now, as a common reference technology we consider the production possibility set $\widetilde{T}$ determined as the free disposal convex hull of the extended sample $\widetilde{Z}$, i.e.,

$$\widetilde{T} = \{(x, y) \mid \sum_{S \subseteq N} \lambda^S x(S) \leq x, \sum_{S \subseteq N} \lambda^S y(S) \geq y, \sum_{S \subseteq N} \lambda^S = 1, \lambda^S \geq 0, \forall S \subseteq N\} \tag{5}$$

Note that using the technology $\widetilde{T}$ does not imply that production plans can be replicated, which is a much stronger assumption of integer constant return to scale that cannot be directly inferred from observing the particular production plans of the (original) sample $Z$.

The free disposal convex hull of the extended sample, $\widetilde{T}$, can be compared to the usual reference technologies of DEA associated with the original sample $Z$; assuming either variable or constant returns to scale (VRS or CRS) respectively:

$$T^{VRS} = \{(x,y) \mid \sum_{j\in N} \lambda^j x^j \leq x, \sum_{j\in N} \lambda^j y^j \geq y, \sum_{j\in N} \lambda^j = 1, \lambda^j \geq 0, \forall j \in N\}, \tag{6}$$

and

$$T^{CRS} = \{(x,y) \mid \sum_{j\in N} \lambda^j x^j \leq x, \sum_{j\in N} \lambda^j y^j \geq y, \lambda^j \geq 0, \forall j \in N\} \tag{7}$$

**Observation 1:** *We note that* $T^{VRS} \subseteq \widetilde{T} \subseteq T^{CRS}$.

Proof: Straightforward and therefore omitted.

In DEA, the efficiency of a specific production plan $z$ relative to a technology set $T$, is typically measured using Farrell's radial index of technical efficiency (Farrell, 1957):

$$E^F_{in}(z,T) = \min\{\theta \mid (\theta x, y) \in T\} \in [0,1]$$

in case of input-orientation, or

$$E^F_{out}(z,T) = \min\{\theta \mid (x, y/\theta) \in T\} \in [0,1]$$

in case of output-orientation.

Note that if $E^F(z,T) = 1$, production plan $z$ is located on an external facet of the convex polyhedral $T$. Ideally, efficiency score 1 should only be assigned to fully efficient production plans (that is, undominated plans in $T$), but it is a well known fact that dominated (frontier) plans may get Farrell score 1 as well; see, e.g., Russell (1985).

In particular, let

$$N^{MPSS} = \{j \in N \mid E^F_{in}(z^j, T^{CRS}) = 1\} \subseteq N^E \tag{8}$$

denote the set of *Most Productive Scale Size* (MPSS) units, i.e., production units in the original sample which are efficient under the assumption of constant returns to scale, see e.g., Cooper et al (2007).

**Observation 2:** *For a given (possibly merged) production plan $z \in \widetilde{T}$, we note that*

$$E^F(z, \widetilde{T}) \geq E^F(z, T^{CRS}),$$

*for both the input and output orientation. Moreover, in case $E^F(z, T^{VRS})$ is well defined we have that*

$$E^F(z, T^{VRS}) \geq E^F(z, \widetilde{T})$$

*for both the input and output orientation.*

Proof: Straightforward consequence of Observation 1.

Now, consider the reduced technology $T_{-j}$ obtained by excluding unit $j$'s production plan from the sample $Z$, defined as,

$$T_{-j} = \{(x, y) \mid \sum_{i \in N \setminus \{j\}} \lambda^i x^i \leq x, \sum_{i \in N \setminus \{j\}} \lambda^i y^i \geq y, \lambda^i \geq 0, \forall i \in N \setminus \{j\}\},$$

with or without the additional convexity requirement $\sum_{i \in N \setminus \{j\}} \lambda^i = 1$.

Using for instance the Farrell index to measure (input oriented) *super efficiency* of production unit $j$ results in the following index,

$$E_{in}^{super}(z^j, T_{-j}) = \min\{\theta \mid (\theta x, y) \in T_{-j}\} \quad (9)$$

If $j$ is a frontier unit the index value will be $\geq 1$ whereas $E_{in}^{super}(z^j, T_{-j}) = E_{in}^F(z^j, T) \in (0, 1)$ for non-frontier units. In particular, note that frontier units which are either convex combinations of fully efficient plans in $Z$ or dominated by those will receive super efficiency score 1, while undominated production plans receive super efficiency scores strictly larger than 1. In this way the super efficiency index (9) induces a complete ranking of the production units in $N$ when (9) is well defined for all units.

There are at least two obvious problems with this approach. First, (9) is not well defined for all units unless constant returns to scale is assumed.[3]

[3]Alternative super efficiency indexes may not even be well defined for all units under constant returns to scale, see e.g., Bogetoft and Hougaard (2004).

Second, while inefficient plans are all compared to the same frontier of the technology, the super efficiency scores of the efficient units are measured against different reference technologies (or different benchmarks so to speak). Obviously this inconsistency makes the relevance of a direct comparison of two super efficiency scores somewhat questionable. Super efficiency scores have therefore also been suggested for alternative purposes, for example, outlier detection, see e.g., Banker and Chang (2006).

**Example 1:** Consider three production units $\{1, 2, 3\}$ using 1 input to produce 1 output with the sample given as $Z = \{z^1, z^2, z^3\} = \{(1, 1), (2, 5), (3, 4)\}$. Then, the extended sample becomes:

$$\widetilde{Z} = \{z(1), z(2), z(3), z(\{1, 2\}), z(\{1, 3\}), z(\{2, 3\}), z(\{1, 2, 3\})\} =$$

$$\{(1, 1), (2, 5), (3, 4), (3, 6), (4, 5), (5, 9), (6, 10)\}.$$

Using $\widetilde{T}$ as reference technology we get the following input oriented Farrell efficiency scores;

$E^F_{in}(z^j, \widetilde{T}) = 1$ for $j \in \{1, 2, \{2, 3\}, \{1, 2, 3\}\}$, $E^F_{in}(z^3, \widetilde{T}) = 7/12$, $E^F_{in}(z^{\{1,2\}}, \widetilde{T}) = 11/12$ and $E^F_{in}(z^{\{1,3\}}, \widetilde{T}) = 1/2$.

Thus, only unit 3 and mergers $\{1, 2\}$ and $\{1, 3\}$ are dominated, while the production plans of all other units and mergers are (undominated) frontier production plans in $\widetilde{T}$.

Using the super efficiency index (9) we get;

$$E^{super}_{in}(z^1, T^{VRS}) = 2, \quad E^{super}_{in}(z^2, T^{VRS}) = inf,$$

and

$$E^{super}_{in}(z^3, T^{VRS}) = E^F_{in}(z^3, T^{VRS}) = 7/12.$$

Thus, we see that the super efficiency index is not always well defined (as in the case of unit 2) and results in the following (incomplete) ranking of the original units; $1 \succ 3$.

# 3 Ranking Production Plans

In the following we shall focus on *input* oriented efficiency[4] and propose two canonical indexes for consistent ranking of production units in $N$, in the sense that efficiency is measured relative to a common reference technology, i.e., the technology (5) estimated on the basis of the extended sample $\widetilde{Z}$.

For all possible mergers of production units $S \subseteq N$, we define the input efficiency of the merged unit $S$ relative to the technology estimated from the *extended* sample $\widetilde{T}$, given by (5), as:

$$e(S) = E^F_{in}(z(S), \widetilde{T}) \in (0, 1] \tag{10}$$

with $e(\emptyset) = 0$ per definition. In other words, $e(S)$ can be seen as the Farrell efficiency score of the merged unit $S$ relative to (5), and can consequently take values between 0 and 1.

**Observation 3:** *We note that;*

1. $e(j) = 1$ *for all* $j \in N^E$.

2. $e(N) = 1$.

3. *There may exist* $S \subseteq N^{MPSS}$ *for which* $e(S) < 1$.

Proof: 1. and 2. are obvious; 3 is proved by Example 3 below. Q.E.D.

For a given production unit $j \in N$, define the *marginal efficiency contribution* of $j$ to the merger $S \subseteq N \setminus \{j\}$ as,

$$m_j(S) = e(S \cup j) - e(S) \ \in \ (-1, 1] \tag{11}$$

That is, $m_j(S)$ indicates how the efficiency of the merged unit $S$ changes by adding production unit $j$ to $S$ relative to the extended technology $\widetilde{T}$. Note that $m_j(S)$ may take any value between -1 and 1, depending on $j$'s influence on the efficiency of $S$. A negative marginal contribution can be the result, for instance, of adding an inefficient unit to an efficient merger, but even efficient units may have negative marginal impact on inefficient

[4]The approach can be used with respect to output oriented efficiency with the obvious changes.

mergers, see e.g., Example 1, where unit 1 is efficient, unit 3 is inefficient and $m_1(3) = e(3 \cup 1) - e(3) = 1/2 - 7/12 = -1/12$.[5]

## 3.1 Contribution Indexes

The marginal efficiency contribution can be seen as a proxy for unit $j$'s contribution to (or influence on) the performance of the merger $S \cup j$. Thus, perhaps the most straightforward parallel to the traditional type of super efficiency index is to rank a unit $j \in N$ by its contribution to the efficiency of $N$, i.e., the marginal efficiency contribution of adding unit $j$ to the merger of the complement $N \setminus \{j\}$.

**Definition:** *We define the marginal efficiency contribution index, $I^M$, of production unit $j \in N$ relative to the technology of the extended sample $\widetilde{T}$ as,*

$$I^M(z^j, \widetilde{T}) = m_j(N \setminus \{j\}). \tag{12}$$

Since $e(N) = 1$ (because no merger can dominate the coalition with maximum output) we get that $I^M(z^j, \widetilde{T}) \in [0, 1)$ for all $j \in N$. Moreover, note that although this index is in some sense a parallel to the traditional super efficiency index it cannot be interpreted as such. Indeed, the unit we measure is not super efficient relative to the frontier of the technology $\widetilde{T}$ based on the extended sample. Thus, we refer to this type of index as a *contribution* index rather than a super efficiency index.

Since there are $2^{n-1}$ possible mergers of $N \setminus \{j\}$ we may also look at the average contribution of unit $j \in N$ over all these mergers as an indication of unit $j$'s average contribution to the efficiency in the sample.

**Definition:** *We define the average efficiency contribution index, $I^A$, of production unit $j \in N$ relative to the technology of the extended sample $\widetilde{T}$ as,*

$$I^A(z^j, \widetilde{T}) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{j\}} m_j(S). \tag{13}$$

[5] In fact, we may even have that the marginal efficiency contribution of an MPSS unit to a unit which it dominates is strictly negative.

Note that we may have negative index values even for efficient units $j \in N^E$ using (13), unlike in the case of the marginal contributions index (12).

The efficiency contribution indexes (12) and (13) induce a ranking of the production units $j \in N$ such that $j$ *outranks* $j'$ (written $j \succ j'$) if and only if $I(z^j, \widetilde{T}) > I(z^{j'}, \widetilde{T})$ for any pair $j, j' \in N$. Clearly, we obtain a complete order with $j \sim j'$ if and only if $I(z^j, \widetilde{T}) = I(z^{j'}, \widetilde{T})$.

**Observation 4:** *We note that;*

*i. Unlike traditional super efficiency indexes, the efficiency contribution indexes (12) and (13) are well defined for all production plans $j \in N$ under weaker assumptions than CRS.*

*ii. Unlike traditional super efficiency indexes, the efficiency contribution indexes (12) and (13) use the same reference technology $\widetilde{T}$ for all production plans $j \in N$.*

*iii. Unlike traditional super efficiency indexes, the efficiency contribution indexes (12) and (13) may rank inefficient units above efficient units (albeit respecting dominance, see Proposition 7 below).*

Proof: The first two statements follow directly from the definitions of (12) and (13). For *iii.*, see e.g., Example 2 below.

Since ranking by contribution indexes (12) and (13) is no longer directly linked to the notion of the (weakly) efficient frontier of the reference technology we may, as observed in *iii.* above, rank inefficient units above frontier units. Yet, it is important to note that both indexes (12) and (13) respects dominance in the sense that if a unit $j$ dominates unit $i$ then $j$ can never be ranked below $i$, see Proposition 7. In other words, an inefficient unit can only be ranked above a frontier unit if the two units are not ordered by the dominance relation (i.e., $j \notin U(i)$ and $i \notin U(j)$). As mentioned in the Introduction, we find this an advantage of the indexes (12) and (13) since not all frontier units are supported by reasonable (optimal) weights and some non-frontier units may in fact be very close to efficiency as well as optimal scale size.

**Remark:** Note that $(N, e)$ where $e$ is given by (10) can be interpreted as a cooperative game. For example, if we redefine (10) as follows;

$$\bar{e}(S) = \begin{cases} 1 & \text{if } E_{in}^F(z(S), \widetilde{T}) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

for all $S \subseteq N$, with $\bar{e}(\emptyset) = 0$ per definition, then $(N, \bar{e})$ is a *simple game* and the associated average contribution index (13) is known as the (normalized) Banzhaf index, see e.g., Peleg and Sudhölter (2003).

# 4 Examples

**Example 1 continued:** Recall the numerical Example 1. First, consider efficient units $N^E = \{1, 2\}$. Starting out with the marginal index $I^M$ we get; $I^M(z^1, \widetilde{T}) = m_1(\{2, 3\}) = 1 - 1 = 0$ and $I^M(z^2, \widetilde{T}) = m_2(\{1, 3\}) = 1 - 0.5 = 0.5$. So clearly, unit 2 outranks unit 1, according to (12).

Next, considering the average index $I^A$ we get; $I^A(z^1, \widetilde{T}) = \frac{1}{4}(m_1(\{2, 3\}) + m_1(2) + m_1(3) + m_1(\emptyset)) = \frac{1}{4}(0 - 1/12 - 1/12 + 1) = 5/24 = 0.2083$ and $I^A(z^2, \widetilde{T}) = \frac{1}{4}(m_2(\{1, 3\}) + m_2(1) + m_2(3) + m_2(\emptyset)) = \frac{1}{4}(1/2 - 1/12 + 1/3 + 1) = 7/16 = 0.4375$. So also according to (13) we have that unit 2 outranks unit 1. Note that the standard super efficiency index of unit 1 is quite high ($E_{in}^{super}(z^1, \widetilde{T}) = 2$) so unit 1 has a considerable impact on the spanning of the frontier of $\widetilde{T}$, but little impact on the efficiency of all the possible mergers.

Finally, for (the inefficient) unit 3 we get: $I^M(z^3, \widetilde{T}) = 0$ and $I^A(z^3, \widetilde{T}) = \frac{1}{4}(0 - 0.5 + 0 + 1) = 1/8 = 0.125$. Thus, $I^M$ induces the ranking $2 \succ 1 \sim 3$ while $I^A$ induces the ranking $2 \succ 1 \succ 3$.

**Example 2:** Consider four production units $\{1, 2, 3, 4\}$ using 1 input to produce 1 output with the sample given as

$$Z = \{z^1, z^2, z^3, z^4\} = \{(1.5, 1), (2.5, 4), (3, 5), (4, 4.5)\}.$$

Then, extended sample becomes:

$$\widetilde{Z} = \{z(1), z(2), z(3), z(4), z(\{1, 2\}), z(\{1, 3\}), z(\{1, 4\}), z(\{2, 3\}), z(\{2, 4\}), z(\{3, 4\}),$$
$$z(\{1, 2, 3\}), z(\{1, 2, 4\}), z(\{1, 3, 4\}), z(\{2, 3, 4\}), z(\{1, 2, 3, 4\})\} =$$

$$\{(1.5,1),(2.5,4),(3,5),(4,4.5),(4,5),(4.5,6),(5.5,5.5),(5.5,9),$$
$$(6.5,8.5),(7,9.5),(7,10),(8,9.5),(8.5,10.5),(9.5,13.5),(11,14.5)\}.$$

Using $\widetilde{T}$ as reference technology we get that $N^E = \{1,2,3\}$ and the super efficiency scores are;

$E_{in}^{super}(z^1,\widetilde{T}) = 5/3$, $E_{in}^{super}(z^2,\widetilde{T}) = 21/20$ and $E_{in}^{super}(z^3,\widetilde{T}) = inf$. Finally, $E_{in}^{F}(z^4,\widetilde{T}) = 11/16$.

We now find index values $I^M$ and $I^A$ for the units 1,2,3 and 4:

$I^M(z^1,\widetilde{T}) = 0.0000$
$I^M(z^2,\widetilde{T}) = 0.1961$
$I^M(z^3,\widetilde{T}) = 0.2569$
$I^M(z^4,\widetilde{T}) = 0.0873$

$I^A(z^1,\widetilde{T}) = 0.1732$
$I^A(z^2,\widetilde{T}) = 0.2988$
$I^A(z^3,\widetilde{T}) = 0.3347$
$I^A(z^4,\widetilde{T}) = 0.0410$

We clearly see that whereas the traditional super efficiency index rank unit 1 above unit 2, both our contribution indexes rank oppositely unit 2 above unit 1, which is more compelling in this case since unit 2 is MPSS (in a 1-input-1-output model) and unit 1 is not.

Moreover, note that using the marginal contributions index $I^M$ we rank the inefficient unit 4 above the efficient unit 1 simply because it has a slightly positive marginal contribution to its complement unlike unit 1. Note though that unit 4 is *not* dominated by unit 1 (cf., Proposition 7 below).

The average index $I^A$ induces perhaps a more reasonable ranking here, namely; $3 \succ 2 \succ 1 \succ 4$.

**Example 3:** Consider a sample consisting of four units $\{1,2,3,4\}$ each using 2 inputs to produce 1 output:

$z^1 = (x_1,x_2,y) = (1,5,1)$
$z^2 = (x_1,x_2,y) = (5,1,1)$

$z^3 = (x_1, x_2, y) = (2, 2, 1)$
$z^4 = (x_1, x_2, y) = (2.2, 2.2, 1)$

Note that unit 4 is inefficient while units 1,2 and 3 are all MPSS units. Clearly, units 1 and 2 are MPSS units because they are "specialized" in their use of input 1 and 2 respectively.

Consequently, the traditional super efficiency approach favors units 1 and 2 that both get super efficiency score 2 while unit 3 only gets super efficiency score 1.1. The induced ranking is therefore: $1 \sim 2 \succ 3 \succ 4$.

Since unit 1 and 2 are specialized units and unit 4 is rather close to being efficient ($E^F_{in}(z^4, \widetilde{T}) = 0.91$) it is far from obvious that unit 4 should be ranked below units 1 and 2. Indeed the more compelling ranking:

$$3 \succ 4 \succ 1 \sim 2$$

is the result of using both types of contribution indexes, $I^M$ and $I^A$ since

$I^M(z^1, \widetilde{T}) = 0.0000$
$I^M(z^2, \widetilde{T}) = 0.0000$
$I^M(z^3, \widetilde{T}) = 0.1220$
$I^M(z^4, \widetilde{T}) = 0.1000$

$I^A(z^1, \widetilde{T}) = 0.0711$
$I^A(z^2, \widetilde{T}) = 0.0711$
$I^A(z^3, \widetilde{T}) = 0.1880$
$I^A(z^4, \widetilde{T}) = 0.1484$

Furthermore, note that merging MPSS units does not guarantee efficiency of the merged unit since e.g., $e(\{1, 2\}) = 0.7$ and $e(\{1, 2, 3\}) = 0.9$.

**Example 4: see Adler et al., (2002).** In order to compare the induced ranking of the proposed contribution indexes to a wider range of existing approaches, we now revisit the illustrative nursing home example from Section 9 in Adler et al., (2002), which is itself an elaboration of an example from Sexton et al., (1986). There are six units (nursing homes) each using two inputs (staff hours per day, $x_1$, and supplies per day, $x_2$) to produce two outputs (total Medicare plus Medicaid reimbursed patient days, $y_1$, and total private patient days, $y_2$). The data set is given as:

$z^1 = (x_1, x_2, y_1, y_2) = (150, 0.2, 14000, 3500)$
$z^2 = (x_1, x_2, y_1, y_2) = (400, 0.7, 14000, 21000)$
$z^3 = (x_1, x_2, y_1, y_2) = (320, 1.2, 42000, 10500)$
$z^4 = (x_1, x_2, y_1, y_2) = (520, 2.0, 28000, 42000)$
$z^5 = (x_1, x_2, y_1, y_2) = (350, 1.2, 19000, 25000)$
$z^6 = (x_1, x_2, y_1, y_2) = (320, 0.7, 14000, 15000)$

The induced ranking of various methods are provided in Table 1, below.

| DEA methods | Induced ranking |
|---|---|
| Cross-efficiency (Sexton et al., 1986) | $1 \succ 2 \succ 4 \succ 5 \succ 3 \succ 6$ |
| Trad. super efficiency (Andersen and Petersen, 1993) | $1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$ |
| Statistical methods | |
| CCA (Friedman and Sinuany-Stern, 1997) | $1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$ |
| DR/DEA (Sinuany-Stern and Friedman, 1998) | $1 \succ 3 \succ 4 \succ 5 \succ 2 \succ 6$ |
| MCDM methods | |
| Maximin eff. ratio (Troutt, 1995) | $4 \succ 3 \succ 5 \succ 1 \succ 2 \succ 6$ |
| MOLP-minimax (Li and Reeves, 1999) | $1 \sim 4 \succ 5 \succ 2 \succ 3 \succ 6$ |
| Contribution indexes | |
| $I^M$ | $5 \succ 1 \sim 2 \sim 3 \sim 4 \sim 6$ |
| $I^A$ | $1 \succ 2 \succ 4 \succ 5 \succ 3 \succ 6$ |

**Table 1:** *Rankings of various methods.*

The rankings of the first six methods are discussed in Adler et al. (2002). Here, we shall briefly note that $I^M$ produces a large indifference class containing all but unit 5 whereas $I^A$ produces a strict ranking. Moreover, the ranking induced by $I^A$ coincides with that of the Cross-efficiency method (using an "aggressive" secondary objective[6]). The cross-efficiency method is designed to address the issue of price- versus technical efficiency and thereby it also aims to free the ranking from a direct connection to the (technical) frontier.

[6] See Sexton et al. (1986).

## 5 Some properties

We start out by showing that both contribution indexes are invariant to rescaling of inputs and outputs $x \mapsto \alpha x$ and $y \mapsto \beta y$ for some strictly positive scalar vectors $\alpha \in \mathbf{R}^s_{++}$ and $\beta \in \mathbf{R}^t_{++}$.

**Proposition 5:** *The contribution indexes $I^M$ and $I^A$ are scale invariant.*

Proof: Scale invariance of $I^M$ and $I^A$ follows from scale invariance of $m_j(S)$, which follows from scale invariance of the Farrell index, see, e.g., Hougaard and Keiding (1998). Q.E.D.

Further, small errors in the measurement of input quantities should only result in small errors in the index value and not lead to dramatic changes in the ranking. Hence, continuity[7] of the ranking index is an important property. Next, we can show,

**Proposition 6:** *The contribution indexes $I^M$ and $I^A$ are continuous (for sets of strictly positive production plans).*

Proof: Both $I^M$ and $I^A$ are defined by the marginal contributions, $m_j(S)$, which in turn are defined as the difference between two Farrell indexes. Hence, the result follows from the fact that the Farrell index satisfies joint continuity in input and technology (on the sub-domain of strictly positive production plans). For details, see Russell (1990). Q.E.D.

Finally, we say that an index, $I$, *respects dominance* if $z' \in U(z)$ implies that $I(z', \widetilde{T}) \geq I(z, \widetilde{T})$. That is, if a unit $z'$ dominates another unit $z$ it can never be ranked below $z$ in the ranking induced by the index.

**Proposition 7:** *The contribution indexes $I^M$ and $I^A$ respect dominance.*

Proof: Consider two units $i$ and $j$ in $N$ for which $j \in U(i)$. First we show that $I^M(z^j, \widetilde{T}) \geq I^M(z^i, \widetilde{T})$. Indeed, we have $e(N \setminus j) \leq e(N \setminus i)$ since $z(N \setminus i) \in U(z(N \setminus j))$ and the Farrell index satisfies weak monotonicity in inputs, see e.g., Russell (1985).

[7]with respect to the topology $\mathcal{T}^\times = \mathcal{T}_E \times \mathcal{T}_C$ where $\mathcal{T}_E$ is the Euclidean topology and $\mathcal{T}_C$ is the topology on technologies induced by the topology of closed convergence.

Second we show that $m_j(S) \geq m_i(S)$ for all $S \subseteq N \setminus \{i, j\}$. Indeed, $e(S \cup j) \geq e(S \cup i)$ for all $S \subseteq N \setminus \{i, j\}$ since $z(S \cup j) \in U(z(S \cup i))$ and the Farrell index satisfies weak monotonicity in inputs. Consequently, $I^A(z^j, \widetilde{T}) \geq I^A(z^i, \widetilde{T})$. Q.E.D.

From Proposition 5-7 we can conclude that, apart from the property of "weak indication",[8] our contribution indexes inherit all the central properties of the Farrell efficiency index. We specifically want to avoid "weak indication" since a Farrell super efficiency index can never rank an inefficient unit above a frontier unit, unlike our indexes. As mentioned, from a ranking perspective, we find it questionable that an almost efficient unit close to being MPSS should be deemed to be ranked below a highly specialized frontier unit, far from optimal scale as is the consequence of "weak indication".

# 6 Final Remarks

The DEA literature contains several approaches to rank production units based on their efficiency with the so-called super efficiency index introduced by Andersen and Petersen (1993) as the best known and most commonly used method. Ranking based on super efficiency indexes has important drawbacks though, and the present paper presents a new approach based on efficiency contribution indexes, which are designed to overcome these drawbacks.

While the marginal contributions index $I^M$ is easy to compute even for large data sets, the average contributions index, $I^A$, is computationally heavier. Indeed, as mentioned in the introduction, $I^A$ resembles the Shapley value and it is well known that determining the Shapley value is computationally complex, see e.g., Bilbao (2000). Unfortunately, $I^A$ inherits this complexity.

Yet, when making pairwise comparisons of the average index $I^A$ we do not need to calculate the efficiency score for each of the $2^n$ possible mergers of the $n$ units. Indeed, looking only at the difference between, say units $i$ and $j$, we need to find the efficiency score of $2^{n-1}$ possible mergers since we can disregard those containing both $i$ and $j$ as well as those that contain neither $i$ nor $j$ (of which there are $2 \cdot 2^{n-2} = 2^{n-1}$). This reduces the number of relevant

[8] Stating that weak efficiency should be indicated by the index value - or specifically for the standard Farrell index: $E^F(z^j, T) = 1$ iff $z^j$ is a frontier unit, see e.g. Russell (1985).

mergers to half the size. Thus, with current computer capabilities and DEA-software without limits on the number of DMUs, most "normal-sized" data sets do not present a serious obstacle for application.

A straightforward extension of our approach is to use the construction of the contribution indexes with respect to a disaggregation of the radial (Farrell) efficiency score into input (or output) specific efficiency scores. For example, using the input specific scores behind the potential improvements index of Bogetoft and Hougaard (1999),(2004), or any other non-radial index such as the Russell or Zieschang index, see e.g., Christensen et al (1999). In this way we would obtain input specific marginal efficiency contributions for each production unit which enables a ranking of these units according to separate inputs.

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