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Abstract

This paper explores a multi-criteria outranking methodology that is designed to both handle uncertain and imprecise data in describing alternatives as well as treating the decision maker's preference information in a sensible way that reflects the difficulties in articulating preferences. Based on fuzzy interval degrees, representing and measuring data imprecision, this procedure obtains a set of semi-equivalence classes assigning an intransitive order on the alternatives. Relevance measures are then explored for ranking alternatives with respect to the semi-equivalence classes, and a final illustrative example is given for comparison with standard methods like PROMETHEE.

The proposed methodology takes into account the risk attitudes of decision makers, organizing the alternatives and ranking them according to their relevance. The whole interactive decision support allows understanding the dependencies among the alternatives and how they can be resolved if a finer ranking is preferred.

Keywords: Weighted overlap dominance, fuzzy interval data, imprecision measures, semi-equivalence classes, relevance ranking

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1. Introduction

Modern information technology allows businesses to collect tremendous amounts of data, and the ability to utilize these for making better decisions is becoming a major competitive factor for all businesses alike. Detailed data makes it possible to harvest the many small gains, for example, by treating customers more individually (e.g. more tailored contents) or by small scale production optimization (e.g. precision agriculture). In parallel the development of creative business models and processes to rationalize large data sets, creates the foundation for more automated decisions and decision support.

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At the heart of this process is basic decision systems and methodologies that match alternatives and preferences. On one side the increasing amount of data requires decision support methods that can handle large uncertain data - data that also by nature, may be imprecise like information extracted from text and images. On the other side, the applied decision support methods should treat the decision maker's preference information in a sensible way such that the resulting decision support (or automated decisions) reflects the uncertainty and imprecision on both sides of this basic matching problem. This includes simple ways to articulate abstract preferences as well as sensible ways to interpret these.

The decision methodology presented in this paper aims at balancing all of these concerns. Alternatives are described by interval data which reflect the natural data imprecision and simple preference articulation such as criteria weights are used to outrank alternatives by pairwise comparison of alternatives. The suggested method does not necessary produce a strictly ranked list of alternatives both rather groups of alternatives reflecting that no clear ranking exists. On the other hand, as data become better and/or the articulation of preferences becomes more precise, the method provides a more strict ranking.

There is a rich literature on outranking methods in multi-criteria decision making; see e.g., [2, 31, 33, 35, 41]. Most methods relates to crisp data, but can be extended to handle interval data as well; such methods include TOPSIS [7], PROMETHEE [28], ELECTRE [44] or VIKOR [37]. Other methods are designed directly for handling interval data such as WOD [27]. Given that data takes the form of intervals there are many advantages of using the latter type of method as argued, e.g., in [19, 20, 27].

In the present paper we revisit the WOD method and address some important issues that were left unexplored by the original paper [27]. In particular, we will be focussing on how to embed the interval information (which is the data input of WOD) into a fuzzy framework based on imprecision measures. We motivate this issue both empirically by examples, and theoretically by associating our approach with ignorance measures as used in the fuzzy literature [6, 36].

Specifically, we argue that data imprecision will always be present in practice; either through multiplicity of data meanings, uncertainty of data measurement, or, multiplicity of modeling approaches. We show that imprecision measures (as in [19, 20]) captures this well. Moreover, we show that there is an intimate connection between our imprecision measures and ignorance functions as analyzed in [6, 36].

Furthermore, we emphasize that the WOD indifference relation is intransitive and that WOD consequently produces a set of semi-equivalence classes with latent dependencies. The dependency, which may arise between some alternatives, is in some sense a natural consequence of modeling most realistic choice situations by the use of intervals. Yet, it may imply that the resulting order of the alternatives in WOD is less operational for specific ranking purposes. We therefore suggest the possibility of refining the outranking order by subsequent use of relevance measures, extending the original proposal of [21].

In particular, we argue that relevance measures are well suited for ranking objects based on their pairwise outranking relations, since the latter lead to a ranking which is finer than that of the semi-equivalence classes resulting from WOD. Furthermore, they offer a general framework to explore decision-making and obtain specific rankings on the set of alternatives, where different examples for relevance measures can be used as the means to resolve in a weak or strict manner, the dependencies among (intransitive) indifferent alternatives.

Finally, we illustrate our relevance-extended WOD framework by comparison with the fuzzy interval version of PROMETHEE [28]. This example allows examining the flexibility and reliability of our proposal, where the outranking order can be refined according to the preferences of the decision maker, identifying the worst and best alternatives, but also understanding their particular dependencies, adding to the interpretability of the final results.

The paper is organized as follows. Section 2 introduces the basic problem, where we justify our approach both from empirical and theoretical standpoints. Section 3 examines the WOD procedure and its outranking order on the alternatives, resulting in semi-equivalence classes which are explored in detail in Section 4. In Section 5, relevance measures are introduced, comparing their behavior with other ranking procedures, such as the non-dominance approach [32], and with respect to the semi-equivalence classes. Our proposal is summarized by an algorithm for learning the relevance ranking of alternatives, following up with the PROMETHEE numerical example in Section 6, and ending with some final remarks and future lines of research.

2. The Problem

We consider a class of multiple criteria decision problems involving a finite (non-empty) set of alternatives $N = \{1, \ldots, n\}$ where each alternative is evaluated with respect to a finite (non-empty) set of criteria $M = \{1, \ldots, m\}$.

For a given criterion $j \in M$, each alternative $a \in N$ is represented by the degree with which a fulfills criterion j measured on the (normalized) valuation scale $L = [0, 1]$. Specifically, we assume that there exists a mapping

$$\mu : N \to \mathcal{L}, \tag{1}$$

where $\mathcal{L}$ is the set of all closed sub-intervals in $[0, 1]$, i.e.,

$$\mathcal{L} = \left\{ x = \left[x^L, x^U\right] \mid \left(x^L, x^U\right) \in [0,1]^2 \text{ and } x^L \leq x^U \right\}.$$

Thus, we consider the case where there is an interval-valued fuzzy set (see [5, 22, 24, 47]) representing the property of *satisfying criterion* $j \in M$, given by,

$$Z_j = \{\langle a, \mu_{Z_j}(a)\rangle \mid a \in N\} \tag{2}$$

where the interval membership degree of each alternative $a \in N$,

$$\mu_{Z_j}(a) = \left[\mu_{aj}^L, \mu_{aj}^U\right], \tag{3}$$

expresses the extent up to which the predicate "a satisfies the property Z_j" can be verified by means of a lower and an upper bound given by μ_{aj}^L and μ_{aj}^U respectively. The relevance of this type of interval data will be motivated both empirically and theoretically below.

Now, taking into account the full set of criteria M, the data for every alternative $a \in N$ is given by m-dimensional cubes

$$c_a = \left[\mu_{a1}^L, \mu_{a1}^U\right] \times \cdots \times \left[\mu_{am}^L, \mu_{am}^U\right], \tag{4}$$

resulting in the full $(n \times m)$ data matrix,

$$\begin{pmatrix} \left[\mu_{11}^L, \mu_{11}^U\right] & \dots & \left[\mu_{1m}^L, \mu_{1m}^U\right] \\ \vdots & \vdots & \vdots \\ \left[\mu_{n1}^L, \mu_{n1}^U\right] & \dots & \left[\mu_{nm}^L, \mu_{nm}^U\right] \end{pmatrix}. \tag{5}$$

The following examples will demonstrate the empirical relevance of our set-up.

2.1. Examples of data imprecision

Interval membership degrees as in (3) represent a very general and flexible way of expressing data uncertainty well suited for many empirical applications. For example, consider the problem of finding the optimal geographical location for a renewable energy (e.g. biogas) plant as in [19].

Here, candidate sites (the alternatives) $a \in N$ consist of geocells of 1 km^2. These cells represent regions which have to be evaluated and ranked based on several criteria $j \in M$ such as, for instance, *proximity to the biomass source.*

Available information includes the road distance (x) and the time (y) it takes for transporting trucks to cover that distance, going from each site to the biomass source. As such, the degree of proximity can be estimated by the expression x/y (where variables x and y are normalized, such that $x, y \in [0, 1]$). At closer inspection, though, variables like x and y are clearly not crisp valued.

For instance, x is related to spatial information since different roads simultaneously pass through the same site, and different paths exist connecting each site with the biomass source. On top of that, regulation implies that specific roads are banned for trucks transporting biomass depending on the daily traffic situation. The resulting uncertainty regarding the value of x is naturally represented by a pair of lower (x^L) and upper (x^U) bounds, such that the degree of proximity can be estimated for each site by the interval $[x^L/y, x^U/y]$.

Now, even if we ignore the uncertainty on x, e.g. by taking the minimum distance existing between any point contained in the site and the biomass source, there may still be uncertainty involved with the transportation time y. The measurement of y depends on many factors, such as the type of truck, the surface of the road, the weight of the load, or even the weather conditions. Hence, we can never be sure that every possible factor of relevance is being considered, and there will always be some level of imprecision left.[1] Consequently, by taking the minimum and maximum time spans for trucks to cover distance x, respectively given by y^L and y^U, the degree of proximity can be imprecisely expressed by the interval $[x/y^U, x/y^L]$.

We can even consider a simpler scenario, where only distances are known, and where a unique value is given for every site regarding its distance to the biomass source. Then, considering the set X of all distances $x \in X$, the proximity property can be modeled by some strictly decreasing function $f : X \to [0, 1]$, such that the more distant, the lesser the proximity of the alternative to the biomass source.

Taking such a function as some type of negation η, we can find several options for modeling the proximity property. One possibility is to take the standard negation (in fact a strong negation, being strictly decreasing, continuous and *involutive*, i.e., where $\eta(\eta(x)) = x$ holds for all $x \in X$), given by,

$$\eta_1(x) = 1 - x, \tag{6}$$

or a strict but not strong negation, such as,

$$\eta_2(x) = 1 - x^2, \tag{7}$$

or lastly, the Sugeno strong negation[2] defined as,

$$\eta_3(x) = \frac{1 - x}{1 + x}. \tag{8}$$

Hence, different models for representing *proximity* can be established, by using either type of negation (6)-(8), such that an interval fuzzy set emerges in case there is no consensus of which particular negation function to use (see Fig. 1, where lower and upper bounds μ^L and μ^U are shown for site a and its distance x_a to the biomass source). That is, an interval degree is obtained (in the sense of (3)), given for every $a \in N$ by the lower bound $\mu^L_{aj} = \eta_3(x_a)$ and the upper bound $\mu^U_{aj} = \eta_2(x_a)$.[3] See e.g. [26] where a similar type of approach is taken to transform various modeling approaches to measure technical efficiency of production units into fuzzy numbers.

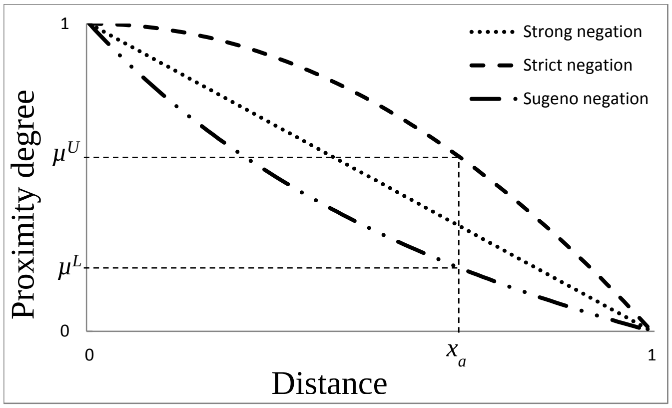
Consequently, *imprecision* occurs at many levels, and its representation in interval form allows managing and exploiting the available information according to its uncertain attributes. Thus, as complexity grows, interval fuzzy sets can be developed into *type-2* fuzzy sets [42, 48, 49], offering a general framework for building the meaning of words and perceptions. In the following we formalize our approach to imprecision and fuzzy interval data.

[1] Notice here that we do not refer to the nature or origin explaining the existence of imprecision, but we just refer to it as a given (quality) phenomenon of uncertainty.

[2] For $\lambda = 1$, see [18, 40]

[3] Notice that we could also use one unique function such as (6), in which case we would have a *precise* measurement given by $\mu = \eta_1(x)$, but this depends on the particular design for membership function μ_{Z_j}, and the meaning of Z_j, according to (1)-(3).

Figure 1: Interval fuzzy set representing the property of *proximity*.



2.2. Characterization of imprecision

Above we identified at least three different sources of data uncertainty in practice: the multiplicity of data meanings, the uncertainty in data measurement, and the multiplicity of modeling approaches. Under a common headline we will refer to such data uncertainties as *imprecision*; initially, without going into further details about its nature or the reasons for its existence.[4]

As a way to express imprecision we suggest to use *imprecision measures* as defined below (see also [19, 20]).

Definition 1. For any membership degree $\mu \in \mathcal{L}$, with respective lower and upper bounds μ^L and μ^U, and for some $\varepsilon > 0$, the function $\delta : [0,1]^2 \to [\varepsilon, 1]$ is an *imprecision measure* if and only if it fulfills the following axioms:

1. $\delta(\mu) = \varepsilon$ if and only if $\mu^U = \mu^L$.
2. $\delta(\mu) = 1$ if and only if $\mu^L = 0$ and $\mu^U = 1$.
3. If $\mu^U > \mu^L$ then $\delta(\mu) > \varepsilon$.
4. For any pair of membership degrees $\mu, \nu \in \mathcal{L}$, $\delta(\mu) > \delta(\nu)$ if and only if $(\mu^U - \mu^L) > (\nu^U - \nu^L)$.

The first axiom states that imprecision is minimal, equal to ε, if and only if the lower and upper bounds are the same. In this sense, imprecision is always present in any type of measurement or perception. The second axiom states that imprecision is maximum if and only if the lower bound is minimal (equal to 0) and the upper bound is maximum (equal to 1). The third axiom says that imprecision receives a positive value greater than ε, every time the upper bound of the interval is greater than its lower bound. Finally, the fourth axiom says

[4]For instance, imprecise measurements, represented by a set of values with lower and upper bounds, can be the result of *ignorance* [6, 8, 9, 36]; *hesitation* [1, 14]; *possibility* and *necessity* [15, 45]; *belief* and *plausibility* [13, 39]; or different confidence levels, like pessimistic and optimistic scenarios, for some estimation.

that the greater the difference between the lower and upper bounds, the greater the imprecision.

Now, it is easy to show that, for instance, the function

$$\delta(\mu) = \max\{\mu^U - \mu^L, \varepsilon\} \tag{9}$$

is an imprecision measure, ensuring non-degenerate intervals even if there were a possibility of having crisp data (i.e., $\mu^L = \mu^U$) from the outset.

In general, imprecision measures can be related to some type of *inequality* metric [12], specifically defined for a given interval degree μ. It will be natural to require that the imprecision should be the same whether we look at μ or the negation of μ. We can show that this is indeed the case for any strong negation[5] η, with (strong) interval negation $\eta^I(\mu)$ being defined for any $\mu \in \mathcal{L}$, as

$$\eta^I(\mu) = [\eta(\mu^U), \eta(\mu^L)]. \tag{10}$$

Proposition 2. *For any interval membership degree $\mu \in \mathcal{L}$, given an imprecision measure δ and a strong interval negation η^I, it holds that $\delta(\mu) = \delta(\eta^I(\mu))$.*

Proof. Following Definition 1, the first axiom states that $\delta(\mu) = \varepsilon$ holds if and only if $\mu^U = \mu^L$. Given that $\eta(\mu^U) = \eta(\mu^L)$ is true if and only if $\mu^U = \mu^L$, it follows that $\delta(\eta^I(\mu)) = \varepsilon$ if and only if $\mu^U = \mu^L$. For the second axiom, stating that $\delta(\mu) = 1$ holds if and only if $\mu^L = 0$ and $\mu^U = 1$ hold, we know that $\eta(\mu^L) = 1$ if and only if $\mu^L = 0$, and that $\eta(\mu^U) = 0$ if and only if $\mu^U = 1$, from where it follows that $\delta(\eta^I(\mu)) = 1$ if and only if $\mu^L = 0$ and $\mu^U = 1$. For the third and fourth axioms, due to the involutive character of η, it is true that $\eta(\mu^L) > \eta(\mu^U)$ if and only if $\mu^U > \mu^L$ holds, and that $\eta(\mu^L) - \eta(\mu^U) = \mu^U - \mu^L$. Hence it follows that if $\mu^U > \mu^L$ then $\delta(\eta^I(\mu)) > \varepsilon$, and for any $\mu, \nu \in \mathcal{L}([0,1])$, it is true that $\delta(\eta^I(\mu)) > \delta(\eta^I(\nu))$ if and only if $(\mu^U - \mu^L) > (\nu^U - \nu^L)$. □

2.3. Imprecision due to ignorance

To be a bit more specific and provide an example of how imprecision could be interpreted theoretically, we shall here examine the relation to *ignorance* in the sense of lack of knowledge for assigning a precise and uniquely valued membership degree $\mu^* \in L$.[6]

In particular, we consider *ignorance functions* [6, 36], modeling the uncertainty due to lack of knowledge regarding the precise membership degree μ^* of a given object to the fuzzy set representing some property of interest. As such, ignorance functions are defined as a continuous mapping $g : [0,1] \to [0,1]$, such that maximum ignorance (i.e., $g(\mu^*) = 1$) prevails when $\mu^* = 0.5$, representing

[5] Recall that a strong negation is strictly decreasing, continuous and involutive

[6] Notice that although ignorance here refers to the knowledge of the limit values of intervals, it should not be confused with rational choice modeling of ignorance [8, 9], as the latter refers to ignorance on the occurrence of events (and the absence of subjective probabilities).

complete lack of knowledge, i.e., a middle state of undecidability between the binary (certain) references $\{0, 1\}$. Besides, the value of ignorance is the same for both the membership degree and its complement (given by the negation η), indicating a close relationship with imprecision measures according to Proposition 2.

A straightforward example of an ignorance function is (see [36]),

$$\tilde{g}(\mu^*) = 2 \min\{\mu^*, \eta(\mu^*)\}. \tag{11}$$

As a result, every time that there is presence of ignorance, an interval degree $\mu \in \mathcal{L}$ is obtained such that the length of the interval is proportional to its amount of ignorance. In fact, for any precise membership degree $\mu^* \in L$ and for multiplicative aggregation operators T and S, such that for any $x, y \in [0, 1]$, $T(x, y) = xy$ and $S(x, y) = x+y-xy$ (see e.g. [18, 38]), the amount of ignorance is equivalent to the length of the interval only if the following holds (the proof for the general result is given in [3]),

$$\mu = [T(\mu^*, \eta(g(\mu^*))), S(\mu^*, g(\mu^*))] \tag{12}$$

Proposition 3. *For any membership degree $\mu^* \in L$ and for any interval membership degree $\mu \in \mathcal{L}$ constructed according to (12), the imprecision measure $\delta(\mu)$ given in (9), coincides with the amount of ignorance $g(\mu^*)$, if and only if $\mu^* \in (0, 1)$.*

Proof. We know that the amount of ignorance associated to the membership degree μ^* coincides with the imprecision of the interval degree μ, only if μ is built according to (12). Hence, under such conditions, and for any value of $\mu^* \in (0, 1)$ such that $g(\mu^*) \neq 0$, it holds that $g(\mu^*) = \mu^U - \mu^L$. On the other hand, we know that $g(\mu^*) = 0$ if and only if μ^* agrees with any of the binary references {0,1} representing certainty, in which case it holds that $g(\mu^*) < \delta(\mu) = \varepsilon$. Then, it follows that if and only if $\mu^* \in (0, 1)$, it holds true that $\delta(\mu) = \max\{(\mu^U - \mu^L), \varepsilon\} = \mu^U - \mu^L = g(\mu^*)$. □

According to the result of Proposition 3, imprecision and ignorance agree with the same value for any $\mu^* \in (0, 1)$, under the construction method of (12). The relation between imprecision and ignorance also suggests that the treatment of imprecise knowledge requires pertinent interval-based reasoning procedures, in order to build reliable decision support.

In the following Section we will explore a particular decision support method based on interval data: the so-called Weighted Overlap Dominance (WOD) multi-criteria procedure initially proposed in [27]. WOD will be examined under our fuzzy framework leading to binary preference relations between pairs of interval valued alternatives.

3. The WOD procedure

Based on the data matrix (5) the decision maker (DM) faces the problem of ordering the alternatives often with the purpose of making a final unique selection of the most preferred alternative. For that use the literature offers several well known (interactive) outranking methods, see e.g., [31, 33, 41]. In the present paper we shall argue that the method dubbed Weighted Overlap Dominance (WOD) in [27], is particularly well suited for handling multi-dimensional interval data. In fact, to our knowledge WOD is the only outranking method that is directly designed to handle interval data. Well known methods like, TOPSIS, ELECTRE, PROMETHEE and VIKOR are all designed for crisp data although they can be extended to handle interval data as well. Yet, it makes a crucial difference since extending for instance the popular TOPSIS method to interval data suffers from weaknesses that are a direct result of the method design itself. For example, in TOPSIS [7, 16], the outranking between any two alternatives depend on their closeness to an ideal and an anti-ideal point, but these points are defined on the basis of the entire data set (as the infeasible min and max of all the data). Therefore the outranking relation between any two alternatives will depend on the data imprecision/uncertainty of an (irrelevant) third alternative. For further discussion, see [27].

In short, WOD relies on a previous specification of criteria weights $w \in \mathbb{R}_m^+$, computing the weighted interval scores and the volume of imprecision for every alternative, and obtains binary *outranking* ($\succ$) or *indifference* ($\sim$) relations according to the the type of overlap between pairs of alternatives.

Let $\mu_a^L = [\mu_{a1}^L, ..., \mu_{am}^L]$ and $\mu_a^U = [\mu_{a1}^U, ..., \mu_{am}^U]$, and consider two alternatives a and b for which $w \cdot \mu_a^U \geq w \cdot \mu_b^U$. If we have that $w \cdot \mu_a^L > w \cdot \mu_b^U$, then there is no overlap and a clearly outranks b. However, if $w \cdot \mu_b^U > w \cdot \mu_a^L$, there is some (volume) overlap and the outranking relation needs to be examined in more detail.

For doing so, define the sets,

$$\hat{Z} = \left\{ i \in c_a | w \cdot i > w \cdot \mu_b^U \right\}, \tag{13}$$

$$\check{Z} = \left\{ i \in c_a | w \cdot i < w \cdot \mu_b^L \right\}, \tag{14}$$

$$\tilde{Z} = \left\{ i \in c_b | w \cdot i > w \cdot \mu_a^L \right\}, \tag{15}$$

where $\hat{Z}$ consists of all the vectors in the data cube of alternative a, c_a, for which the value is higher than the maximal value of alternative b (note that $\hat{Z} = \emptyset$ only if $w \cdot \mu_a^U = w \cdot \mu_b^U$); $\check{Z}$ consists of all the vectors in the data cube of alternative a, c_a, for which the value is lower than the minimal value of alternative b; and $\tilde{Z}$ consists of all the vectors in the data cube of alternative b, c_b, for which the value is higher than the minimal value of alternative a (note that $\tilde{Z} = \emptyset$ only if $w \cdot \mu_a^L > w \cdot \mu_b^U$).

Now, if $\check{Z} = \emptyset$ there is partial volume overlap in the sense that $w \cdot \mu_a^L \geq w \cdot \mu_b^L$, such that

$$a \succ b \;\Leftrightarrow\; P(a,b) > \beta, \tag{16}$$

where $\beta \in [0,1]$ is a user determined parameter and

$$P(a,b) = \frac{V(\hat{Z})}{V(c_a)} + \frac{V(c_a \setminus \hat{Z})}{V(c_a)} \frac{V(c_b \setminus \tilde{Z})}{V(c_b)}, \tag{17}$$

with $V(\cdot)$ being the imprecision volume operator defined for every $a \in N$ by,

$$V(c_a) = \delta(\mu_{Z_1}(a)) \times ... \times \delta(\mu_{Z_m}(a)). \tag{18}$$

The idea is that P is a proxy for the probability that a vector drawn from the data cube c_a has a higher value than a vector drawn from the data cube c_b, and this likelihood has to be higher than β in order for a to outrank b. In this sense, β is a risk attitude parameter for establishing the outranking of a over b, where the higher it is, the lower the risk.

If $P(a,b) \leq \beta$, both alternatives are considered indifferent, i.e., $a \sim b$ holds.

On the other hand, if $\check{Z} \neq \emptyset$ there is complete volume overlap in the sense that c_a contains c_b, such that $w \cdot \mu_a^L \geq w \cdot \mu_b^L$ and $w \cdot \mu_a^U \leq w \cdot \mu_b^U$. In this case, it holds that

$$a \succ b \;\Leftrightarrow\; C(a,b) > \gamma, \tag{19}$$

for (user determined) parameter $\gamma \in \mathbb{R}_+$ and

$$C(a,b) = \frac{V(\hat{Z})}{V(\check{Z})}. \tag{20}$$

Similarly, if $C(a,b) = \gamma$ or $C(a,b) < \gamma$ it respectively holds that $a \sim b$ or that $b \succ a$.

The idea here is that the outranking situation is determined by the ratio between those vectors in c_a having higher value than any vector in c_b and those having lower value than any vector in c_b. Hence, γ expresses the risk attitude towards outranking a over b, such that the higher it is, the lower the risk.

It is straightforward to show that, for every $a, b \in N$, the WOD outranking relation $\succ$ is *asymmetric*, i.e, if $a \succ b$ holds, then $b \succ a$ does not hold. In [27] it is shown that the outranking situation $\succ$ is semi-transitive (i.e., $a \succ b, b \succ c \not\Rightarrow c \succ a$). Moreover, we can show the following result.

Proposition 4. *The WOD indifference relation $\sim$ is a dependency relation, i.e., $\sim$ is reflexive, symmetric and intransitive.*

Proof. It is straightforward to show that the relation $\sim$ is reflexive, such that for every $a \in N$, $a \sim a$ holds, and symmetric, such that for every $a, b \in N$, $a \sim b \Leftrightarrow b \sim a$. Now, regarding its intransitivity, we have that for every $a, b, c \in N$, if $a \succ b$ and $b \succ c$ hold, it cannot hold that $c \succ a$, due to the semi-transitivity of $\succ$. Therefore, either $c \sim a$ or $a \succ c$ hold. By contradiction, assume that $\sim$ is transitive. Then, if it holds that $c \sim a$ and $b \succ c$, it should also hold that $b \succ a$, contradicting the asymmetry of $\succ$. Consequently, $\sim$ is intransitive. □

Therefore, the aggregated order for the whole set of alternatives may contain particular dependencies which will be further explained in the next section; an aspect that was left unexplored in the paper [27].

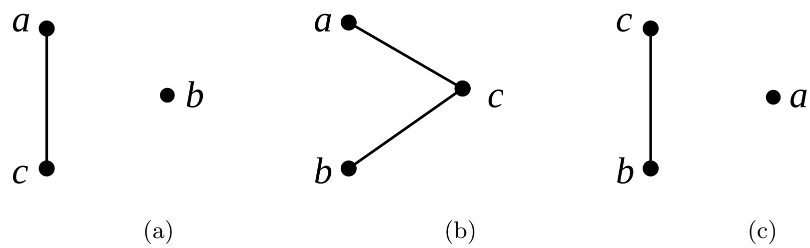


Figure 2: Relational graph where a higher alternative (node) outranks a lower one and two alternatives are connected (by an edge) if an indifference situation holds between them.

4. Semi-equivalence classes

The intransitivity of the indifference relation is a characteristic attribute of the WOD outranking procedure, representing realistic decision situations, despite the consequent loss of desirable mathematical properties. In fact, the intransitivity is not surprising, given the large number of examples in preference literature having intransitive indifference due to their interval numerical representation [34]. According to the multi-criteria interval degrees of the alternatives, indifference reveals a certain dependency due to the *similarity* or *proximity* among alternatives, which can naturally be intransitive (see e.g. [29]).

The non-transitivity of the indifference relation $\sim$ in WOD gives rise to semi-equivalence classes, that can formally be defined as follows.

Definition 5. A subset of alternatives $E \subseteq N$ is a semi-equivalence class if and only if its elements are either related through the indifference relation $\sim$, or in case $a \succ b$ for some pair $a, b \in E$ then a depends on b via a third element $c \in E$, such that either $a \sim c$ and $c \not\succ b$ or $c \succ a$ and $b \sim c$.

The three possibilities for dependency among alternatives $a, b, c \in N$, where $a \succ b$ hold, are shown in Fig.2. Definition 5 suggests an implicit hierarchy or linear order among different semi-equivalence classes given by the specific pairwise relations of the alternatives. Let Ξ denote the set of semi-equivalence classes, such that for any two semi-equivalence classes $E, E' \in \Xi$, we say that E is superior to E' (written as $E \rhd E'$) if and only if $a \succ b$ for any $a \in E$ and $b \in E'$. Such a hierarchy emerges naturally as shown in the following result.

Proposition 6. *For any pair of semi-equivalence classes $E, E' \in \Xi$, and for any pair of alternatives $a \in E$ and $b \in E'$, it holds that $E \rhd E'$ if and only if there is no other element $c \in N$ such that $a \not\succ c$ and $c \not\succ b$.*

Proof. From Definition 5, we know that if $a \in E$ and $b \in E'$, then $a \succ b$, and that there is no $c \in N$ such that $a \sim c$ and $c \not\succ b$ or that $c \succ a$ and $b \sim c$. Then, the only possibility left for a dependency situation to hold between a and b, and hence for $E = E'$ to hold, is for both $c \succ a$ and $b \succ c$ to hold true, which is impossible due to the semi-transitivity of $\succ$. On the other hand, if for any $a \in E$ and $b \in E'$ there does not exist a $c \in N$ such that $a \not\succ c$ and $c \not\succ b$, then it is straightforward that $E \rhd E'$. □

The existence of semi-equivalence classes highlights the complexity of assigning a *weak order* or ranking to all the alternatives based on their outranking relations in WOD. In fact, it is not always possible to establish such an order unless some additional specific rank-construction procedure is subsequently applied.

For operational reasons, decision makers often prefer a finer ranking than that offered by semi-equivalence classes, since these may be quite large in practice. We suggest to employ some form of *relevance-based ranking* method. In the next section we explore such methods regarding the semi-equivalence classes, as well as examining other procedures found in literature regarding decision making purposes [32].

5. Ranking alternatives according to their relevance

Exploring the ranking of alternatives, such that for any pair of alternatives, one can be placed either higher, lower or in the same level as the other, a relevance-based approach is presented here based on *relevance degrees* [21].[7] We also consider a non-dominance (relevance) ranking based on [32], and examine its behavior with respect to the semi-equivalent classes.

5.1. Relevance measures

Relevance measures can be defined for the WOD procedure in order to organize the alternatives according to their relative importance. The basic idea is to assign a relevance value to every alternative, increasing with the number of alternatives that it outranks and the importance of the outranked alternatives. So, estimating the alternatives' relevance allows grading them on a linear scale, such that a ranking is automatically obtained.

Relevance measures are characterized as follows.

Definition 7. For every $a, b, c, d \in N$, where D_a represents the set of all alternatives that are outranked by a, the function $\sigma : N \to [0, 1]$ is a relevance measure if and only if the following hold:

1. $\sigma(a) = 0$ if and only if $D_a = \emptyset$.
2. If $a \succ b$ and $D_b = \emptyset$, then $\sigma(a) > \sigma(b)$.
3. If $D_a = \{N \setminus \{a\}\}$, then for every $b \neq a \in N$, $\sigma(a) > \sigma(b)$.
4. If $a \succ b$, $d \succ c$ and $b \succ c$, then $\sigma(a) > \sigma(d)$.

Definition 7 characterizes relevance regarding the relative importance of alternatives, examining the conditions for minimum and maximum relevance, and its general behavior regarding outranked alternatives. Such conditions state that (1) if an alternative does not outrank any other alternative, it has no relevance at all; (2) an alternative gains relevance with the number and the importance

[7]Initially conceived for measuring the relative importance of alternatives according to the verification of chains of preference relations.

of the alternatives that it outranks; (3) if an alternative outranks every other alternative, its relevance is maximal; (4) if two alternatives outrank the same number of alternatives, the one that outranks more important alternatives obtains higher relevance.

The proposition below suggests a straightforward (and operational) candidate for a relevance measure.

Proposition 8. *For every $a \in N$, let $d_a = |D_a|$ be the number of elements in D_a. Then, the function given by*

$$\tilde{\sigma}(a) = d_a + \sum_{k \in D_a} d_k \tag{21}$$

is a relevance measure.

Proof. In order to check if the function $\tilde{\sigma}$ is a relevance measure, we have to see if it fulfills the conditions in Definition 7. First, it can be seen that if $D_a = \emptyset$ holds, then a does not outrank any alternative, and its importance value d_a is null. Hence, its relevance is also null until it outranks at least one alternative. Only then, when it outranks at least one alternative, its relevance will be greater than 0, fulfilling the second condition. Third, if a outranks every other alternative in N, this means that the value of d_a is greater than any other value d_b where $b \neq a \in N$, and that $\sum_{k \in D_a} d_k > \sum_{k \in D_b} d_k$, such that $\tilde{\sigma}(a) > \tilde{\sigma}(b)$. And finally, if any two alternatives $a, b \in N$ outrank the same number of alternatives, such that $d_a = d_b$, then their difference in relevance depends on the alternatives they outrank, such that $\tilde{\sigma}(a) > \tilde{\sigma}(b)$ holds only if it also holds that $\sum_{k \in D_a} d_k > \sum_{k \in D_b} d_k$. □

Example 9. Consider four alternatives $a, b, c, d \in N$, and the following (outranking) relations:

$$a \succ b,\ a \succ c,\ a \succ d,$$
$$b \sim c,\ b \sim d,\ c \succ d.$$

Alternative a outranks alternatives b, c and d, while the alternatives b and c are indifferent. The indifference between b and c is somewhat problematic since b is indifferent to d, but at the same time c outranks d. In other words, N is partitioned into two semi-equivalence classes $E \cup E' = N$ where $E = \{a\}$ and $E' = \{b, c, d\}$.

Now, applying the relevance measure $\tilde{\sigma}$, we obtain that,

$$\tilde{\sigma}(a) = 4,\ \tilde{\sigma}(b) = 0,\ \tilde{\sigma}(c) = 1,\ \tilde{\sigma}(d) = 0,$$

and as a result we obtain a finer ranking than that offered by the semi-equivalence classes: a is the first alternative of the ranking, followed by c and then by the two indifferent alternatives b and d.

Example 9 illustrates how a ranking of alternatives can be obtained by a particular relevance measure. We now show that such a ranking based any relevance measure in the sense of Definition 7, respects the induced hierarchy of the semi-equivalence classes.

Proposition 10. *For every $a, b \in N$, if $a \in E$ and $b \in E'$, where $E \rhd E'$, then $\sigma(a) > \sigma(b)$.*

Proof. According to Proposition 6, any member of E dominates any other member of a lower ranking equivalence class E'. Then, if $a \in E$ and $b \in E'$, it holds that $a \succ b$, where it holds that $D_b \subseteq D_a$. Hence, a outranks at least one alternative more than b, such that $D_b^* = \{D_b \setminus D_b \cap D_a\} = \emptyset$ and $D_a^* = \{D_a \setminus D_b \cap D_a\} \neq \emptyset$, from where it follows, from Definition 7, Axiom 2, that $\sigma(a) > \sigma(b)$ holds. □

Consequently, relevance measures allow exploiting the outranking information extracted from the WOD procedure, refining it into a strict ranking, in the form of a totally ordered set of semi-equivalence classes Ξ, without modifying the existing outranking order.

5.2. Relevance and the non dominance ranking

The non dominance technique is a well-known decision-making method in the preference literature [18, 32]. It consists in ranking the alternatives according to their non dominance degree, expressing the extent up to which an alternative is dominated by no other alternative in N.

Under the WOD framework, for every $a \in N$, the non dominance degree $ND(a)$ can be obtained from the expression,

$$ND(a) = \max_{b \in D_a^{\succeq}} \{d_b\}, \tag{22}$$

where $D_a^{\succeq}$ denotes the subset of alternatives that are outranked by, or indifferent with, a.

Here, by examining (22) under the relevance framework, we can gain a deeper understanding on the ranking problem regarding a given set of semi-equivalence classes. Therefore, the non dominance technique can be explored under the relevance framework by considering the *relevance-non dominance* degree, consisting in the computation for all $a \in N$, of the following expression,

$$\sigma_{ND}(a) = \max_{b \in D_a^{\succeq}} \{\sigma(b)\}, \tag{23}$$

where alternatives are ranked according to their relevance-non dominance scores.

Proposition 11. *For every $a, b \in N$ and for any pair of equivalence classes $E \rhd E' \in \Xi$, if $a \in E$ and $b \in E'$, then it holds that $\sigma_{ND}(a) \geq \sigma_{ND}(b)$.*

Proof. From Proposition 10, we know that for every alternative $a \in N$, there exists a relevance value $\sigma(a)$, which is known to respect the hierarchy of Ξ. Hence, given that for every $a \in N$, the value for $\sigma_{ND}(a)$ agrees with the maximal relevance value σ, among alternatives $b \in D_a^{\succeq}$, it will hold that $\sigma_{ND}(a)$ agrees with the value of σ belonging to some lower or equally ranked alternative, preserving the hierarchy of Ξ, but allowing ties between a and b, from where it follows that $\sigma_{ND}(a) \geq \sigma_{ND}(b)$. □

Corollary 12. *For every* $a, b \in N$ *and for any pair of semi-equivalence classes* $E \triangleright E' \in \Xi$*, if* $a \in E$ *and* $b \in E'$*, then it holds that* $ND(a) \geq ND(b)$*.*

Following the previous results, the relevance-non dominance technique *weakly* respects the hierarchy among semi-equivalence classes, given that it allows for ties among alternatives belonging to different classes. In this sense, this technique weakly respects the hierarchy on Ξ, obtaining a finer order than the one assigned by the semi-equivalence classes, but less strict than with relevance measures.

The overall procedure for ranking alternatives according to the hierarchy of semi-equivalence classes is given in Algorithm 1. Then, given the set of equivalence classes Ξ, the outranking order assigned on N can be further refined into a ranking by the relevance-non-dominance technique (22), such that a stricter ranking can be obtained (stricter with respect to Ξ) by means of the relevance ranking (21).

Algorithm 1 Learning the relevance-ranking R of alternatives

Input: Data matrix (5)
Output: Ranking of N
R-1 Identify the set of semi-equivalence classes Ξ, following (13)-(20)
R-2 Obtain a weak Ξ-ranking, based on (23)
R-3 Obtain a strict Ξ-ranking, based on (21)

6. Illustrative example comparing PROMETHEE and WOD procedures

The PROMETHEE framework has been extended to fuzzy and interval frameworks (see [2, 23, 28]), where preference functions determine the value of preference between pairs of alternatives regarding a given criterion. Here, following [28], for every criterion $j \in M$ and alternative $a \in N$, there is a function $f_j(a)$ grading the performance of criterion j for alternative a, which is valued by an interval, such that $f_j(a) = [a^L, a^U]$. Then, the value for preference between alternatives $a, b \in N$,

$$P_j(a,b) = [P_j^L(a,b), P_j^U(a,b)], \tag{24}$$

is given by,

$$P_j(a,b) = [a^L - b^U, a^U - b^L]. \tag{25}$$

In this sense, the fuzzy WOD approach is very similar to that of PROMETHEE. In the former, the design of the property (criteria j) being measured determines the interval score for the alternatives (3), while in the latter, interval data is given by f_j. Nonetheless, the fuzzy WOD proposal offers a solid methodology for treating interval information, as it is its primary focus to deal with imprecise measurements, exploiting interval data according to its imprecision (volume) overlapping, and handling its uncertainty based on the parameters β (17) and γ (20) for establishing pairwise binary outranking relations.

Now, regarding PROMETHEE, an outranking interval degree $\pi(a, b)$ is obtained from the weighted average of all the m values of $P_j(a, b)$. Here, where the WOD obtains crisp outranking relations, PROMETHEE obtains degrees of outranking, using such degrees for computing the *leaving* (φ^+) and *entering* (φ^-) flows for every alternative, respectively expressing the aggregated degree of outranking and inverse outranking, such that

$$\varphi^+(a) = \frac{1}{n-1} \sum_{b \neq a \in N} \pi(a, b), \tag{26}$$

and

$$\varphi^-(a) = \frac{1}{n-1} \sum_{b \neq a \in N} \pi(b, a). \tag{27}$$

Notice that the leaving flow φ^+ is computed from the aggregation of the outranking degrees, in the same way that the relevance ranking (21) computes the importance of an alternative $a \in N$, given by d_a, according to the number of alternatives that it outranks (where WOD outranking either holds or not, as it is not a matter of degree). The problem then, for PROMETHEE, is to rank alternatives according to their leaving, entering or *net flow* (e.g., the net flow can be given by the difference between the leaving and the entering flows), taking into consideration the uncertain relations between different intervals, something that WOD handles from the beginning in order to establish outranking and indifference situations.

Following [28], worst and best scenarios are built by respectively taking only the lower and upper bounds of the leaving flow, transforming (defuzzyfying) the interval information into a real number, thus yielding a *crisp* net flow and ranking alternatives accordingly to the classical PROMETHEE [4]. Hence, the handling of the risk is determined by the particular selection of worst and best scenarios, obtaining a net flow from where the actual ranking develops, without allowing further interaction of the decision maker on how to treat interval uncertainty, as with WOD.

The numerical application found in [28] considers a set of four alternatives $N = \{a, b, c, d\}$, and a set of three equally weighted criteria $M = \{c_1, c_2, c_3\}$, which have to be minimized. The decision matrix is given in Table 1. Both crisp and interval-based PROMETHEE results obtain that alternative d is the most preferred one, followed by b and c which are indifferent between themselves, and lastly a which is the worst one.

Applying the WOD procedure on the data of Table 1, we can examine the results of measuring interval uncertainty by imprecision measures, where the user can interact with the model and its determination of the outranking relations by the risk-attitude parameters β and γ [27].

In this way, by ranking alternatives according to their relevance (21), different results can be obtained regarding β and γ. In this example, the parameter β

Table 1: Decision matrix

	c_1	c_2	c_3
a	[0,13]	[552,801]	[526,1783]
b	[0,12]	[527,811]	[526,1783]
c	[0,13]	[556,805]	[40,171]
d	[0,12]	[531,815]	[40,171]

has a high impact on the results. That is, with high values of β (expressing risk aversion), such that $\beta > 0.6$, all the alternatives in N appear to be indifferent, while for lower values of β (expressing risk propensity), such that $\beta < 0.3$, alternatives c and d are indifferent between themselves, and preferred to a and b, being a indifferent to b. Otherwise, setting both parameters β and γ in neutral risk attitudes, such that $\beta = 0.4$ and $\gamma = 1$, the WOD relevance ranking obtains that alternative d is the best one, followed by all other three alternatives, which are indifferent between themselves (see Fig 3a).

Thus, if the decision maker is risk averse, then all four alternatives are considered to be indifferent, but if he is willing to take a greater risk, then some alternative can possibly outrank another one. This is the case under risk propensity, where alternatives d and c effectively outrank a and b. Otherwise, maintaining a neutral attitude, alternative d is the winner, arriving to similar results as PROMETHEE, but failing to identify that a is the worst option. Nonetheless, following Algorithm 1, and ranking alternatives according to (23), it is then obtained that alternatives d, c, and b are indifferent between themselves but preferred to a, leaving a as the worst alternative (see Fig. 3b).

Hence, the resolution of the intransitivity of the outranking order can be better understood following the WOD decision process. By looking at the different rankings, the decision maker can learn in an interactive way the effects of taking distinct risk attitudes, naturally emerging from interval pairwise comparisons, with the objective of identifying well-suited choices under a coherent treatment of uncertainty. Meanwhile, different types of rankings are built, regarding the existing semi-equivalence classes (as in Algorithm 1), where the decision maker (user) can choose the required level of *refinement* on the ordering of alternatives. In this sense, the whole interactive process of the fuzzy WOD system incorporates the user's knowledge, offering reliable and interpretable decision support.

On the other hand, relevance measures offer a general and robust framework for exploring other specific ranking procedures (as previously done for the non dominance (22) and the relevance-non dominance (23) rankings). For example, based on PROMETHEE's net flow, alternatives could also be ranked according to a *net relevance* value, given for every $a \in N$ by

$$Net_{\sigma}(a) = \sigma(a) - \sum_{b \in D_a^{\prec}} \sigma(b), \tag{28}$$

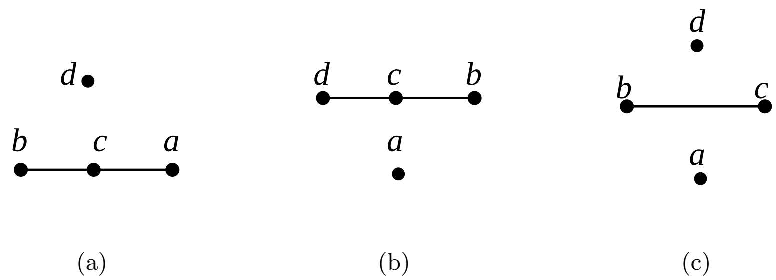


Figure 3: Results with risk neutral attitudes for the (a) strict-Ξ ranking, (b) weak-Ξ ranking and (c) Net$_\sigma$-ranking

where $D_a^{\prec}$ represents the set of alternatives that outrank a. Then, by taking this net-relevance measure which strictly respects the hierarchy among semi-equivalence classes [8], the same ranking as in PROMETHEE is obtained, where alternative d is the better one, followed by the indifferent alternatives b and c, and followed by the worst alternative a (see Fig. 3c).

7. Final remarks

A multi-criteria decision support procedure has been provided for handling imprecise data, with the purpose of ranking alternatives according to their (imprecisely measured) multiple attributes and their estimated relevance. We have explored the complexity of ranking alternatives based on non-transitive indifference and semi-transitive outranking relations, where relevance measures have been proven to offer a reliable and flexible framework for ranking alternatives based on their pairwise preference relations. Overall, the procedure proposed here is useful for decision makers, aiding them to organize and identify the best alternatives, following Algorithm 1. Such a ranking is dependent on the decision maker's risk attitudes, where it is possible to develop a human-system interactive process, examining the different rankings and the risk involved in each one of them.

It remains for further research to explore the dissention-consensus ranking problem [17], for the aggregation of rankings for multiple decision makers (see e.g. [10]). Besides, the WOD decision support system can be enhanced by means of a human-system interactive design for handling different criteria weights and risk-attitudes, even learning the particular risk-types of decision makers, and aggregating rankings according to their associated type.

[8] The net-relevance measure strictly respects the hierarchy of Ξ, given that it always holds, for any $a \in E$ and $b \in E'$, where $E \rhd E'$, that $\sigma(a) > \sigma(b)$ and that $\sum_{k \in D_a^{\prec}} \sigma(k) < \sum_{k \in D_b^{\prec}} \sigma(k)$

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