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Abstract

This paper extends the Weighted Overlap Dominance (WOD) model (initially presented in J.L. Hougaard, K. Nielsen. Weighted Overlap Dominance - A procedure for interactive selection on multidimensional interval data. Applied Mathematical Modelling 35, 2011, 3958 - 3969), as an outranking approach for decision support and multidimensional interval analysis. First, the original approach is extended using fuzzy set theory which makes it possible to handle both non-interval and interval data. Second, we re-examine the ranking procedure based on semi-equivalence classes and suggest a new complementary ranking procedure. The new ranking procedure introduces relevance degrees for ranking the given set of alternatives.

In this way, a complete methodology is presented for identifying recommended solutions that aid the decision-maker when facing a specific problem, by ranking alternatives according to their relevance under imprecise measurements.

Keywords: weighted overlap dominance, outranking, fuzzy data, interval imprecision, relevance ranking

1. Introduction

Recently, [11] introduced a new interactive outranking procedure for ordering a given set of alternatives based on mutidimensional *interval* data, referred to as the Weighted Overlap Dominance (WOD) Method. The WOD method belongs to the general class of outranking methods (see. e.g., [3, 18]) under the important distinction that it is directly built for treating interval information, unlike other outranking procedures that may be extended to do so, see e.g., [4, 12, 19]. Since most methods are constructed for real data, their extensions are highly sensitive to possible outliers as well as the general uncertainty of the data. WOD avoids

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this problem by building on pairwise comparisons of alternatives. Besides, its decision support approach allows the interaction of the decision-maker (DM) through user-defined parameters, which can be tuned up in order to explore their effect over the final ordering of the alternatives.

In this paper, we will present a closer study of two particular aspects of the WOD methodology:

The first concerns a deeper inquiry into the relation between the aggregation methodology used in WOD and the kind of data information which fits the modeling framework. To be more precise, the WOD method develops pairwise comparisons by means of a volume-based aggregation operator. We observe that as information may also take the form of precise data, i.e., of non-interval values, the volume operator needs to be extended so it can give proper account of both precise and imprecise measurements. In this sense, a proper interpretation of the type of uncertainty represented by means of intervals needs to be addressed.

The second aspect concerns the way we can use and interpret the outranking relational situations resulting from WOD. In short, the outranking order resulting from the WOD procedure does not provide a weak order over the set of alternatives. This issue arises from the *semi-transitivity* of its *outranking* relation and the *non-transitivity* of its *indifference* relation.

In order to cope with these issues, we extend the WOD methodology by means of fuzzy set theory [9, 21] (developing the initial approach proposed in [5, 6]), exploring the representation of interval degrees under *uncertainty due to imprecision* and the construction of a ranking of alternatives according to their *relevance*. From this fuzzy perspective (see e.g., [1, 2, 7, 10, 15, 16]), we notice that the uncertainty expressed by intervals can be classified at least under the two general categories of *imprecision* and *ignorance*. The former makes reference to the *quality* of the information and the knowledge that can be inferred from it, while the latter refers to the *quantity* of information and the absence or lack of knowledge. In this sense, this investigation refers to *uncertainty due to imprecision*, understanding interval degrees as reliable measurements built from real-life observations.

Furthermore, we explore the WOD outranking order and suggest to extend its methodology for arriving to a recommended course of action, i.e., to a ranking of alternatives, by means of *relevance degrees* (in the sense of [8]). As proposed in [5, 6], this approach on relevance identifies particular solutions that aid the DM along the learning-decision process.

The paper is organized as follows. First, the initial proposal for the WOD model is reviewed, presenting its basic definitions and methodology, as introduced in [11]. Second, we examine the WOD model from a fuzzy perspective, focusing on the extraction of knowledge from imprecise data. Third, we explore some properties for the WOD outranking and indifference relations, identifying some difficulties that arise if a ranking of alternatives is to be obtained. Fourth, we propose relevance measures for obtaining such a ranking. We close with some final comments and remarks on topics for future research.

2. The WOD procedure

Following the original WOD methodology, as previously presented in [11], consider a set of alternatives $N = \{1, ..., n\}$ and a set of criteria $M = \{1, ..., m\}$, such that for every alternative $a \in N$ and criterium $j \in M$, there is an interval value expressing the score of a with respect to j. Such an interval is given by $\left[x_{aj}^L, x_{aj}^U\right]$ where it holds that $x_{aj}^L \leq x_{aj}^U$.

Input data for the WOD model therefore consist of the m-dimensional cubes

$$c_a = \left[x_{aj}^L, x_{aj}^U\right]^m,$$

containing the scores for every alternative in N according to the m criteria.

Now, let $w \in \mathbb{R}_+^m$ be a vector of weights representing the importance of the criteria in M, and label any pair of the alternatives in such a way that $w \cdot x_a^U \geq w \cdot x_b^U$ holds (relabel if necessary), where $x_a^U = \left(x_{a1}^U, ..., x_{am}^U\right)$. Lastly, let V stand for some volume operator such that $V(\emptyset) = 0$, and define the three sets:

$$\hat{Z} = \left\{i \in c_a | w \cdot i > w \cdot x_b^U\right\},$$

$$\check{Z} = \left\{i \in c_a | w \cdot i < w \cdot x_b^L\right\},$$

and

$$\tilde{Z} = \left\{i \in c_b | w \cdot i > w \cdot x_a^L\right\}.$$

In this way, the WOD procedure obtains the outranking and indifference relational situations for every pair $a, b \in N$, depending on the type of overlap existing between the alternatives' scores. The first type of overlap refers to *strict domination*. Here we say that one cube c_a *strictly dominates* another cube c_b if the inequality $x_a^L > x_b^U$ holds, which is true if $\tilde{Z} = \emptyset$. Hence, if it is true that $\tilde{Z} = \emptyset$, it is said that a outranks b, i.e., it holds that $a \succ b$.

The second type of overlap is the one of *partial overlap*, where it is true that $\check{Z} = \emptyset$. Here, the outranking situation $a \succ b$ holds only if the relation represented by $P(a, b)$ receives a value greater than some $\beta \in [0, 1]$, where

$$P(a, b) = \frac{V\left(\hat{Z}\right)}{V(c_a)} + \frac{V\left(c_a \setminus \hat{Z}\right)}{V(c_a)} \frac{V\left(c_b \setminus \tilde{Z}\right)}{V(c_b)}.$$

Otherwise, if $P(a, b) \leq \beta$, both alternatives are considered indifferent, i.e., $a \sim b$ holds.

Finally, the third type of overlap makes reference to a *complete overlap*, where it holds that $\check{Z} \neq \emptyset$. Then, for some $\gamma \in \mathbb{R}_+$, it is true that $a \succ b$ only if the following inequality holds

$$\frac{V\left(\hat{Z}\right)}{V\left(\check{Z}\right)} > \gamma,$$

it is true that $a \sim b$ only if the following equality holds

$$\frac{V\left(\hat{Z}\right)}{V\left(\check{Z}\right)} = \gamma,$$

and it is true that $b \succ a$ only if the following inequality holds

$$\frac{V\left(\hat{Z}\right)}{V\left(\check{Z}\right)} < \gamma.$$

Therefore, an outranking order is assigned over the alternatives in N, based on the construction of the outranking and indifference relations regarding the user-defined parameters β and γ. These parameters allow learning the different situations rising between the alternatives and their interval scores. Hence, both parameters represent outranking thresholds that under different types of overlapping, allow exploiting the information taking the form of interval data.

Besides, a third parameter $\alpha \in [0, 1]$ is also defined for reducing the set of alternatives that come into the pair-wise analysis. This is done by defining the alternative $a_0 \in N$ as the one with the maximal weighted value, such that

$$a_0 = \arg\max_{a \in N} w \cdot x_a^U,$$

and for every alternative $a \in N$, defining the set

$$c_a | a_0 = \left\{ i \in c_a | w \cdot i \geq w \cdot x_{a_0}^L \right\},$$

representing the weighted overlap between alternatives a and a_0. Then, the DM is able to consider only the alternatives where the ratio between $c_a|a_0$ and c_a allows verifying that

$$\frac{V\left(c_a|a_0\right)}{V\left(c_a\right)} \geq \alpha.$$

In consequence, the WOD procedure suggests a methodology for assigning an outranking order over the α-selected alternatives, depending on the way the DM handles the uncertainty by means of the parameters β and γ. In this way, revising the outcome of the WOD procedure, the DM learns the preference relations that can be built from the interval scores, finding support for taking an effective decision.

For more details on the WOD model, its pair-wise comparison methodology and its approach to decision support, along with its relation to other models, see [11].

3. A fuzzy approach to the WOD procedure

In this Section, the type of uncertainty that is being represented by interval data is examined by means of the *fuzzy-WOD* model, introducing *imprecision measures* and their m-dimensional characterization of the volume operator V.

3.1. Exploring the WOD procedure under interval fuzzy sets

Based on the original WOD procedure, as it has been reviewed in Section 2, a fuzzy approach is now explored for modeling imprecision over the multidimensional interval data (continuing the initial proposal of [6]). This approach consists in applying fuzzy set theory [9, 21] for representing the degree in which the set of alternatives N satisfy the set of criteria M.

In this way, a fuzzy set A is defined over a set of objects of interest N, by

$$A = \{\langle a, \mu_A(a)\rangle \,|a \in N\},$$

where

$$\mu_A(a) : N \to [0,1]$$

is its characteristic function, assigning an element of the valuation scale $L = [0,1]$ to every object in N, and

$$\mu_A(a) \in [0,1]$$

is its membership degree, verifying up to which extent some $a \in N$ satisfies the property represented by A.

Under this setting, a unique value is used for expressing the degree in which A is verified by some $a \in N$, but it is noticed that assigning an exact number to such a degree is not always possible to do. Think for example in the measurement of food quality, where sensorial characteristics like the *tenderness* of meat or the *body* of wine have to be perceived and judged by a panel of experts. In that case, it is natural to identify a set of values or some linguistic label representing the degree in which the property of interest can be verified, without requiring one unique and precise numerical value. Hence, interval-valued fuzzy sets can be used in order to capture the uncertainty of common measurements and perceptions (see [10], but also [1, 2, 15, 17]).

In this sense, an interval-valued fuzzy set Z is characterized by a mapping

$$\mu_Z(a) : N \to L([0,1]),$$

where its membership degree is given by

$$\mu_Z(a) \in L([0,1]),$$

with $L([0,1])$ being the set of all closed sub-intervals in $[0,1]$, such that,

$$L([0,1]) = \left\{ x = \left[x^L, x^U\right] \mid \left(x^L, x^U\right) \in [0,1]^2 \text{ and } x^L \leq x^U \right\}.$$

Therefore, recalling the multi-criteria decision support framework of the WOD procedure, and letting Z_j represent the property of *satisfying criterium* $j \in M$, the interval membership degree

$$\mu_{Z_j}(a) = \left[\mu^L_{Z_j}(a), \mu^U_{Z_j}(a)\right],$$

allows verifying the predicate "a satisfies the property Z_j" in a gradual and general way. Then, taking into account the set of criteria M, the input data for the *fuzzy-WOD* model is given by the m-dimensional cubes

$$c_a = \left[\mu^L_{Z_j}(a), \mu^U_{Z_j}(a)\right]^m.$$

Once the membership interval degrees are given by c_a for every $a \in N$, the sets $\hat{Z}$, $\check{Z}$ and $\tilde{Z}$ can be respectively defined, where

$$\mu^L(a) = \left(\mu^L_{Z_1}(a), ..., \mu^L_{Z_m}(a)\right),$$

$$\mu^U(a) = \left(\mu^U_{Z_1}(a), ..., \mu^U_{Z_m}(a)\right),$$

and

$$w \cdot \mu(a) = \left[w \cdot \mu^L(a), w \cdot \mu^U(a)\right].$$

Now, exploring the original WOD analysis of the multidimensional cubes, a volume operator V is used for evaluating the overlap between the alternatives' degrees. In this way, the magnitude of the intervals can be measured, making reference to some general type of uncertainty. Here, by means of the fuzzy-WOD model, an attempt to deal with the imprecision of interval degrees is given, also extending the procedure for the treatment of data with no definite volume (think for example in the case where the upper and lower bounds of intervals coincide).

Example 1. Consider a simple scenario where membership degrees are defined for two alternatives $a, b \in N$ and one criterium $j = 1$, such that their lower and upper bounds coincide, as in

$$\mu_{Z_1}(a) = [0, 0]$$

and

$$\mu_{Z_1}(b) = [1, 1].$$

Firstly, the WOD procedure requires to relabel the alternatives such that the inequality $\mu^U_{Z_1}(a) > \mu^U_{Z_1}(b)$ holds. As a result, we obtain that

$$\mu_{Z_1}(a) = [1, 1]$$

and

$$\mu_{Z_1}(b) = [0, 0].$$

Then, verifying that $\tilde{Z} = \emptyset$, it holds that $a \succ b$, which is a result that follows common sense.

Example 2. Now consider the case where the lower and upper bounds of the interval degrees coincide between the alternatives, as in

$$\mu_{Z_1}(a) = [0.5, 0.5]$$

and

$$\mu_{Z_1}(b) = [0.5, 0.5].$$

Then, an indifference relation would be expected to hold between a and b. But it is the case that, independently on how the alternatives are labeled, $\tilde{Z} = \emptyset$ holds, and the outranking relation $a \succ b$ holds true.

In consequence, the WOD procedure does not allow to properly solve problems dealing with non-interval degrees, further suggesting its incompleteness for handling both interval and non-interval data, every time the upper bounds of the membership degrees coincide.

Example 3. Consider the following pair of membership degrees, given for two alternatives $a, b \in N$ and one criterium $j = 1$, such that

$$\mu_{Z_1}(a) = [0, 1]$$

and

$$\mu_{Z_1}(b) = [1, 1],$$

where it holds that $\mu^U_{Z_1}(a) = \mu^U_{Z_1}(b)$. Then, two different cases hold according to the WOD procedure. In the first case, maintaining the same labeling of alternatives, it is verified that $\check{Z} \neq \emptyset$. Hence, examining the ratio between the volume of $\hat{Z}$ and the volume of $\check{Z}$, it is obtained that

$$\frac{V\left(\hat{Z}\right)}{V\left(\check{Z}\right)} = 0,$$

given that $V\left(\hat{Z}\right) = V(\emptyset) = 0$, and as a result it holds that $b \succ a$.

In the second case, by relabeling the alternatives such that

$$\mu_{Z_1}(a) = [1, 1]$$

and

$$\mu_{Z_1}(b) = [0, 1],$$

then $\check{Z} = \emptyset$ holds, and it is obtained that

$$P(a, b) = 0 \not> \beta,$$

given that $V\left(\hat{Z}\right) = V(\emptyset) = 0$ and that $V\left(c_a \setminus \hat{Z}\right) = V(\emptyset) = 0$. Therefore, it is verified that an indifference relation holds between a and b, i.e., it holds that $a \sim b$, which contradicts the result of the previous case.

In this way, the initial WOD procedure, being explicitly designed for handling interval data, encounters some difficulties when dealing with non-interval data, and in general, with the special situations where the weighted membership degrees of different alternatives obtain equal upper bounds. This is a problem also related to the nature of the uncertainty that the volume operator V is set out to measure, and the type of knowledge being represented by interval degrees.

3.2. Measuring uncertainty due to imprecision

Exploring the WOD procedure, some difficulties have been pointed out to exist regarding the various forms that knowledge may take under uncertainty. In order to cope with these difficulties, and furthermore, in order to give an explicit description of the type of uncertainty being represented by intervals, we suggest to extend the WOD procedure by means of *imprecision measures* as we define them below (see also [5, 6]).

These type of measures characterize the magnitude of the interval, given by the membership degree μ_{Z_j}, as the uncertainty due to imprecision existing over it.

Definition 4. Let μ_{Z_j} be the membership degree of Z_j, given by the respective lower and upper bounds $\mu^L_{Z_j}$ and $\mu^U_{Z_j}$. Then, for some $\varepsilon \in (0,1)$, the function $\delta : [0,1]^2 \to [\varepsilon, 1+\varepsilon]$ is an *imprecision measure* if and only if it fulfills the following axioms:

1. $\delta\left(\mu_{Z_j}\right) = \varepsilon$ if and only if $\mu^U_{Z_j} = \mu^L_{Z_j}$.
2. $\delta\left(\mu_{Z_j}\right) = 1+\varepsilon$ if and only if $\mu^L_{Z_j} = 0$ and $\mu^U_{Z_j} = 1$.
3. If $\mu^U_{Z_j} > \mu^L_{Z_j}$ then $\delta\left(\mu_{Z_j}\right) > \varepsilon$.
4. For two memebership degrees μ_{Z_j} and ν_{Z_j}, $\delta\left(\mu_{Z_j}\right) > \delta\left(\nu_{Z_j}\right)$ if and only if $\left(\mu^U_{Z_j} - \mu^L_{Z_j}\right) > \left(\nu^U_{Z_j} - \nu^L_{Z_j}\right)$.

The first axiom states that imprecision is minimal, equal to ε, if and only if the lower and upper bounds are the same. In this sense, imprecision is always present in any type of measurement or perception. The second axiom states that imprecision is maximum, and equal to $1+\varepsilon$, if and only if the lower bound is minimal (equal to 0) and the upper bound is maximum (equal to 1). The third axiom says that imprecision receives a positive value greater than ε, every time the upper bound of the interval is greater than its lower bound. Finally, the fourth axiom says that the greater the difference between the lower and upper bounds, the greater the imprecision.

Proposition 5. *The function*

$$\delta^*\left(\mu_{Z_j}\right) = \mu^U_{Z_j} - \mu^L_{Z_j} + \varepsilon$$

is an imprecision measure.

Proof. In order to check if δ^* is an imprecision measure we have to check that it fulfills conditions 1-4 of Definition 4. First, $\delta^*\left(\mu_{Z_j}\right) = \mu^U_{Z_j} - \mu^L_{Z_j} + \varepsilon = \varepsilon$ is true if and only if it holds that $\mu^U_{Z_j} - \mu^L_{Z_j} = 0$, which is only true if it holds that $\mu^U_{Z_j} = \mu^L_{Z_j}$. Second, $\delta^*\left(\mu_{Z_j}\right) = \mu^U_{Z_j} - \mu^L_{Z_j} + \varepsilon = 1+\varepsilon$ is true if and only if it holds that $\mu^U_{Z_j} - \mu^L_{Z_j} = 1$, which is only true if it holds that $\mu^L_{Z_j} = 0$ and $\mu^U_{Z_j} = 1$. Third, if it holds that $\mu^U_{Z_j} > \mu^L_{Z_j}$ then it is true that $\delta^*\left(\mu_{Z_j}\right) = \mu^U_{Z_j} - \mu^L_{Z_j} + \varepsilon > \varepsilon$. And finally, $\delta^*\left(\mu_{Z_j}\right) > \delta^*\left(\nu_{Z_j}\right)$ is true if and only if it holds that $\left(\mu_{Z_j}^U - \mu_{Z_j}^L + \varepsilon\right) > \left(\nu_{Z_j}^U - \nu_{Z_j}^L + \varepsilon\right)$, which is true only if $\left(\mu_{Z_j}^U - \mu_{Z_j}^L\right) > \left(\nu_{Z_j}^U - \nu_{Z_j}^L\right)$. □

In this way, the fuzzy-WOD model defines the volume operator V^f, measuring the magnitude of uncertainty due to imprecision, such that for any $a \in N$,

$$V^f(c_a) = \delta(\mu_{Z_1}) \cdot \delta(\mu_{Z_2}) \cdot \ldots \cdot \delta(\mu_{Z_m}) = \delta^m(c_a).$$

Proposition 6. *For any $a \in N$, the fuzzy-WOD volume operator V^f is an m-dimensional imprecision measure, such that*

$$V^f(c_a) = \delta^m(c_a).$$

Proof. Using a standard volume operator, we have that the volume of one point is not defined, that the volume for an interval is equal to its length l, and for a given number of dimensions m, that the volume of some *hypercube* C is given by $V(C) = l^m$. Assigning the meaning of imprecision to intervals, by means of imprecision measures in the sense of Definition 1, we have that for any $a \in N$, the volume of imprecision for some single point value c_{a_0} is given by $V^f(c_{a_0}) = \delta(c_{a_0}) = \varepsilon$ (Axiom 1, Definition 1), and in the same way, the volume for some interval value c_{a_l} is given by its length l including the volume of the single-point, such that $V^f(c_{a_l}) = l + \varepsilon = \delta(c_{a_l})$ (Axioms 2-4, Definition 1). Generalizing for m dimensions, we have that the volume of imprecision for c_a is given by $V^f(c_a) = (l+\varepsilon)^m = \delta^m(c_a)$. □

Therefore, the original WOD and the fuzzy-WOD models are related under the characterization of the volume-based aggregation operator as an m-dimensional imprecision measure.

Nonetheless, the fuzzy-WOD model includes a new clause on the initial WOD methodology in order to cope with the difficulties illustrated by Examples 2 and 3, but without compromising the overall outranking procedure. Such clause refers to the initial labeling of alternatives, where the WOD procedure [11] states that for any pair of alternatives $a, b \in N$ and for some vector of weights w, the inequality $w \cdot \mu^U(a) \geq w \cdot \mu^U(b)$ holds. Due to the problems rising when the inequality turns into an equality, the following clause is formulated:

For each pair $a, b \in N$, assume that $w \cdot \mu^U(a) > w \cdot \mu^U(b)$ holds, otherwise relabel, and in the case the equality $w \cdot \mu^U(a) = w \cdot \mu^U(b)$ holds, label the alternatives according to their imprecision volume, such that

$$V^f(w \cdot c_a) > V^f(w \cdot c_b),$$

where $w \cdot c_a = w \cdot \mu(a)$. In any other case, if $w \cdot \mu^U(a) = w \cdot \mu^U(b)$ and $V^f(w \cdot c_a) = V^f(w \cdot c_b)$ hold, then both alternatives are indifferent, i.e., it holds that $a \sim b$.

Now, the previous Examples 2 and 3 can be re-examined, illustrating how the fuzzy-WOD model allows handling the previous inconsistencies of the initial WOD procedure.

Example 7. Revisiting Example 2, where the interval degrees between the alternatives a and b are the same, we have that the condition $w \cdot \mu^U(a) = w \cdot \mu^U(b)$ holds. In consequence, the imprecision between them has to be measured, following the clause of the fuzzy-WOD approach. Then, it is obtained that both degrees have the same imprecision, such that

$$V^f(w \cdot c_a) = V^f(w \cdot c_b) = \varepsilon,$$

and as a result, in concordance with common sense, $a \sim b$ holds.

Example 8. Revisiting Example 3, we have that the upper bounds of the interval degrees between alternatives a and b coincide, such that $w \cdot \mu^U(a) = w \cdot \mu^U(b)$. So, their imprecision has to be measured in order to label them accordingly, where it is obtained that

$$V^f(w \cdot c_a) = V^f([0,1]) = 1 + \varepsilon,$$

and

$$V^f(w \cdot c_b) = V^f([1,1]) = \varepsilon.$$

As a result, given that

$$V^f(w \cdot c_a) > V^f(w \cdot c_b),$$

we are in the case of complete overlap, i.e., in the case where $\check{Z} \neq \emptyset$ holds, verifying that

$$\frac{V^f\left(\hat{Z}\right)}{V^f\left(\check{Z}\right)} = 0.$$

Therefore, the outranking relation $b \succ a$ holds, which agrees with common sense.

4. Properties of the WOD outranking and indifference relations

Under both the original and the fuzzy WOD procedures, the set of alternatives is ordered according to the interval input data cubes, the vector of criteria weights w, and the DM's choice of parameters α, β and γ. In this way, two different relations hold for every pair of alternatives $a, b \in N$: either the outranking relation $\succ$, or the indifference relation $\sim$. Here we explore their characteristic mathematical properties, further examining the results of [6, 11] and the type of order that the outranking (fuzzy-)WOD procedure assigns over N.

First of all, it can be shown that the outranking relation $\succ$ is *irreflexive*, such that for every $a \in N$, $a \succ a$ does not hold; and *asymmetric*, such that for every $a, b \in N$, if $a \succ b$ holds, then $b \succ a$ does not hold.

Proposition 9. *For every $a, b \in N$, the fuzzy-WOD outranking relation $\succ$ is irreflexive and asymmetric.*

Proof. In order to check irreflexivity, we have to see if it holds that $a \nsucc a$. In this way, under the fuzzy-WOD procedure, when an alternative is compared with itself, it is verified that $w \cdot \mu^U(a) = w \cdot \mu^U(a)$ and that $V^f(w \cdot c_a) = V^f(w \cdot c_a)$ hold. Then it is true that $a \sim a$, i.e., a can never outrank a. Hence, the fuzzy-WOD outranking relation $\succ$ is irreflexive. On the other hand, checking asymmetry, the fuzzy-WOD procedure labels alternatives such that $w \cdot \mu^U(a) > w \cdot \mu^U(b)$, or if $w \cdot \mu^U(a) = w \cdot \mu^U(b)$ holds, such that $V^f(w \cdot c_a) > V^f(w \cdot c_b)$. Under those terms, it always holds either one of the three situations $a \succ b$, $b \succ a$ or $a \sim b$, such that if $a \succ b$ holds, then it cannot be true that $b \succ a$. □

Now, recalling the most important result of [11], it has been shown that the outranking relation $\succ$ is a semi-transitive relation, such that for every $a, b, c \in N$, if $a \succ b$ and $b \succ c$ hold, then $c \succ a$ does not hold. Hence, following this result, it can be shown that the indifference relation $\sim$ is not an equivalence relation, where mathematical equivalence consists of being reflexive, symmetric and transitive.

Proposition 10. *For every $a, b, c \in N$, the fuzzy-WOD indifference relation $\sim$ is not an equivalence relation.*

Proof. In order to check if $\sim$ is an equivalence relation, we have to see if it is reflexive, symmetric and transitive. Firstly, checking reflexivity, we have that it always holds that $a \sim a$ (see the proof of the previous Proposition 9). Secondly, it is symmetric, given that the comparison between a and b is always done according to the condition that either $w \cdot \mu^U(a) = w \cdot \mu^U(b)$ and $V^f(w \cdot c_a) = V^f(w \cdot c_b)$ hold, that $w \cdot \mu^U(a) = w \cdot \mu^U(b)$ and $V^f(w \cdot c_a) > V^f(w \cdot c_b)$ hold, or that $w \cdot \mu^U(a) > w \cdot \mu^U(b)$ holds. In either of those three cases, if it is true that $a \sim b$ then it is also true that $b \sim a$. Thirdly, checking transitivity, we know that for every $a, b, c \in N$, if $a \succ b$ and $b \succ c$ hold, then it cannot hold that $c \succ a$. Therefore, it is true that either $c \sim a$ or $a \succ c$. So, if we assume that $\sim$ is transitive, then, if it holds that $c \sim a$ and $b \succ c$, it should also hold that $b \succ a$, but this is not true by contradiction. As a result, $\sim$ is not necessarily transitive, and hence, it is not an equivalence relation. □

Rather than being a drawback, the non-transitivity of the indifference relation is a strength of the (fuzzy-)WOD model and its outranking procedure. That is, according to the alternatives and their membership degrees, transitivity may or may not hold, depending on the particular situations existing between the alternatives involved. In this sense, indifference represents a *similarity* or *proximity* relation (which can naturally be non-transitive, as shown in [13]), and in the case it respects transitivity, it represents an *equality* or *indistinguishability* relation (which would indeed be transitive, although regarding indistinguishability, it depends on the way it is formally defined).

Corollary 11. *For every $a, b \in N$, the fuzzy-WOD indifference relation $\sim$ is a dependency relation, i.e., a reflexive and symmetric relation.*

In consequence, the outranking order that the (fuzzy-)WOD procedure assigns over N is built on a semi-transitive outranking relation and a dependent (non-equivalent) indifference relation. This means that it is not always the case that alternatives can be effectively ranked, i.e., it is not always possible to obtain a weak order over N. In this way, as shown in Proposition 4, the (fuzzy-)WOD procedure allows that for some $a, b, c \in N$, the three situations of $a \succ b$, $b \succ c$ and $c \sim a$ hold, where a definite *dependency* exists given that c and a are equally ranked, but one of them is above b and the other one is below b.

The solution taken in [11] assumes the existence of a general type of *semi-equivalence* class, where the alternatives presenting such kind of *dependency* can be categorized. The principal drawback here is that by classifying such alternatives under an *semi-equivalence* class, important pieces of information are lost along the way, in particular those referring to the outranking of alternatives by means of the $\succ$ relation. Even more, by categorizing the dependent alternatives under *semi-equivalence* classes, the decision support model is unable to explain and deal with the decision problem in a descriptively satisfactory way. Nonetheless, assuming *semi-equivalence* over the dependent alternatives allows obtaining a ranking such that for any two alternatives, the first one is either ranked higher, equal or lower than the second one.

In the following section we propose a different approach on the ranking of alternatives by means of the fuzzy-WOD model and the use of *relevance measures* (initially defined in [8]). By doing so, the decision support procedure of the fuzzy-WOD model is able to describe and explain how to rank the alternatives according to their multidimensional interval degrees, the vector of criteria weights w, and the DM's choice of parameters α, β and γ.

5. Ranking alternatives using relevance measures

In this Section, a methodology for ranking the alternatives following the fuzzy-WOD procedure is proposed, based on *relevance measures* [8]. Notice that such a ranking of alternatives may have to be constructed (as it has been noted in the previous Section 4), in order to offer additional support to the DM for understanding the decision problem and its possible solutions. In this sense, the fuzzy-WOD model can be extended in order to build and obtain a ranking over N, also allowing the DM to verify how the choice of different parameters and weights affect such a ranking.

Here, it is observed that every time an alternative outranks another alternative, its importance increases, and at the same time, that such an increment is higher if the outranked alternative is also an important one. This is the basic intuition for the following characterization of relevance, where a *relevance degree* is assigned to every alternative in N according to the quantity and the importance of the alternatives that it outranks.

Definition 12. Given the outranking relation $\succ$, and for every $a, b, c, d \in N$, let $D_a = \{j \in N \mid a \succ j\}$ represent the set of all alternatives that are outranked by a. Then, the function σ is a *relevance measure*, where σ_a represents the *relevance degree* of a, if and only if it fulfills the following axioms:

1. $\sigma_a = 0$ if and only if $D_a = \emptyset$.
2. If $a \succ b$ and $D_b = \emptyset$, then $\sigma_a > \sigma_b$.
3. If $D_a = \{N \setminus \{a\}\}$, then for every $b \neq a \in N$, $\sigma_a > \sigma_b$.
4. If $a \succ b$, $c \succ d$ and $b \succ d$, then $\sigma_a > \sigma_c$.

Relevance measures, as presented in Definition 12, characterize the notion of relevance in the following way. (1) If an alternative does not outrank any other alternative, it has no relevance at all; (2) an alternative gains relevance with the number and the importance of the alternatives that it outranks; (3) if an alternative outranks every other alternative, its relevance is maximum; and (4) one alternative is more relevant than another if it outranks alternatives with a higher degree of importance.

Proposition 13. *For any $a \in N$, let $d_a = |D_a|$ be the number of elements in D_a. Then, the function given by*

$$\sigma^*_a = d_a + \sum_{k \in D_a} d_k$$

is a relevance measure.

Proof. In order to check if the function σ^* is a relevance measure, we have to see if it fulfills condition 1-4 of Definition 12. First, it can be seen that if $D_a = \emptyset$ holds, then a does not outrank any alternative, and its importance value d_a is null. Hence, its relevance is also null until it outranks at least one alternative. Only then, when it outranks at least one alternative, its relevance will be greater than 0, fulfilling also the second axiom. Third, if a outranks every other alternative in N, this means that the value of d_a is greater than any other value d_b where $b \neq a \in N$, and that $\sum_{k \in D_a} d_k > \sum_{k \in D_b} d_k$, such that $\sigma^*_a > \sigma^*_k$. And finally, if any two alternatives $a, b \in N$ outrank the same number of alternatives, such that $d_a = d_b$, then their difference in relevance depends on the alternatives they outrank, such that $\sigma^*_a > \sigma^*_b$ holds only if it also holds that $\sum_{k \in D_a} d_k > \sum_{k \in D_b} d_k$. □

In consequence, the fuzzy-WOD model obtains a ranking over N based on the relevance degrees of the alternatives, where each alternative has an associated degree that is either greater, equal or less than the degree of any other alternative.

Example 14. Consider four alternatives $a, b, c, d \in N$, and the following relations:

$$a \succ b,\ a \succ c,\ a \succ d,$$

$$b \sim c,\ b \sim d,\ c \succ d.$$

From these relational situations, it can be seen that alternative a outranks alternatives b, c and d, and examining alternatives b and c, it can be seen that they are indifferent. Hence, dealing with the problem of ranking these alternatives, the indifference relation between b and c comes into question, due to the fact that b is indifferent to d and at the same time c outranks d. In this way, applying the relevance measure σ^*, we obtain that,

$$\sigma_a^* = 4,\ \sigma_b^* = 0,\ \sigma_c^* = 1,\ \sigma_d^* = 0,$$

and as a result, a is the first alternative of the ranking, followed by c and then by the two indifferent relations of b and d. On the contrary, notice that applying the original WOD solution [11], the three alternatives b, c and d would be declared as *indifferent* alternatives.

Example 14 illustrates how the ranking of alternatives can be obtained according to their relevance. We now note that the proposed relevance ranking respects the original *semi-equivalence* classes.

Let the *semi-equivalence* classes resulting from application of the original WOD method be labeled $E_q, q = \{1, 2, 3, ...\}$, where $E_q \succ E_{q'}$ for $q' > q$.

Proposition 15. *For every $a, b \in N$, if $a \in E_q$ and $b \in E_{q'}$, where $q' > q$, then $\sigma_a > \sigma_b$.*

Proof. By definition any member of E_q dominates any other member of a lower ranking equivalence class $E_{q'}$, where $q' > q$. Then, if $a \in E_q$ and $b \in E_{q'}$, it means that $a \succ b$, where it holds that $D_b \subseteq D_a$. Hence, a outranks at least one alternative more than b, such that $D_b^* = \{D_b \setminus D_b \cap D_a\} = \emptyset$ and $D_a^* = \{D_a \setminus D_b \cap D_a\} \neq \emptyset$, from where it follows, by Axiom 2 of Definition 12, that $\sigma_a > \sigma_b$ holds. □

Therefore, relevance measures allow exploiting the information contained in the original *semi-equivalence* classes, introduced in [11], without modifying the existing outranking relations.

6. Final remarks

An extension of the WOD procedure has been presented under a fuzzy approach, referred to as the fuzzy-WOD model, describing the interval uncertainty by means of imprecision measures, and obtaining a ranking of alternatives based on relevance measures. In this way, the DM is able to learn the effects of choosing a different vector of criteria weights w and parameters α, β and γ, adjusting them to his preferences. This fuzzy approach extends the original WOD proposal [11], offering a general methodology for solving decision problems under imprecision and relevance.

As future lines of research, a decision support system's approach requires the fuzzy-WOD model to be capable of dealing with social decision-making and the aggregation of different experts or individual DMs. The challenge is to combine the preference information following from different choices of decision parameters, where multiple rankings are created, and recommend optimal courses of action based on the group consensus.

Further, it is observed that besides indifference, other situations may arise between the pairs of alternatives where an optimal best decision cannot be identified. That is, in order to make the model capable of handling the distinct situations existing on the decision problem, other relations can be included besides $\succ$ or $\sim$, making reference to some type of *conflict* or *neutrality* over the alternatives.

Finally, it is observed that the fuzzy-WOD model and its application to decision support can be enhanced by allowing the DM to value criteria weights and the free parameters β and γ in linguistic terms. In this sense, it would not be required to find one precise value for making human evaluations, but instead, take advantage of the human ability to handle and evaluate perceptions through language (see e.g. [14, 20]). This is an issue directly related to the interval representation of knowledge and the natural imprecision found in symbols, words and common language.

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[1] K. Atanassov, G. Gargov. Interval valued intuitionistic fuzzy sets. Fuzzy Sets and Systems 31, 343-349 (1989)

[2] E. Barrenechea, A. Fernandez, F. Herrera, H. Bustince. Preference relations using ignorance functions: interval-valued non dominance criterion. Advances in Intelligent and Soft Computing 107, 243-255 (2012)

[3] M. Behzadian, R. Kazemzadeh, A. Albadvi, M. Aghdasi. PROMETHEE: A comprehensive literature review on methodologies and applications. European Journal of Operational Research 200, 198-215 (2010)

[4] T. Chen, Ch. Tsao. The interval-valued fuzzy TOPSIS method and experimental analysis. Fuzzy Sets and Systems 159, 1410-1428 (2008)

[5] C. Franco, M. Bojesen, J.L. Hougaard, K. Nielsen. The Fuzzy WOD Model with application to biogas plant location. 8th International Conference on Soft Computing Models in Industrial and Environmental Applications, Salamanca, September 11-13, 2013.

[6] C. Franco, J.L. Hougaard, K. Nielsen. A fuzzy approach to the Weighted Overlap Dominance Model. 15th Conference of the Spanish Association for Artificial Intelligence, Madrid, September 17-20, 2013.

[7] C. Franco, J.T. Rodríguez, J. Montero. Relational structures for measures of ignorance. Advances in Intelligent Systems Research 1, 613-619 (2011)

[8] C. Franco, J.T. Rodríguez, J. Montero. Relevance in preference structures. Advances in Intelligent Systems and Computing 214 (2013)

[9] J. Goguen. The logic of inexact concepts. Synthese 19, 325-373 (1969)

[10] I. Grattan-Guinness. Fuzzy membership mapped onto interval and many-valued quantities. Mathematical Logic Quarterly 22, 149-160 (1976)

[11] J.L. Hougaard, K. Nielsen. Weighted Overlap Dominance - A procedure for interactive selection on multidimensional interval data. Applied Mathematical Modelling 35, 3958-3969 (2011)

[12] J.F. LeTéno, B. Mareschal. An interval version of PROMETHEE for the comparison of building products' design with ill-defined data on environmental quality. European Journal of Operational Research 109, 522-529 (1998)

[13] R.D. Luce. Semiorders and a theory of utility discrimination. Econometrica 24, 178-191 (1956)

[14] L. Martínez, D. Ruan, F. Herrera. Computing with words in decision support systems: an overview on models and applications. International Journal of Computational Intelligence Systems 3, 382-395 (2010)

[15] J. Montero, D. Gómez, H. Bustince. On the relevance of some families of fuzzy sets. Fuzzy Sets and Systems 158, 2429-2442 (2007)

[16] N. Pal, H. Bustince, M. Pagola, U. Mukherjee, D. Goswami, G. Beliakov. Uncertainties with Atanassov's intuitionistic fuzzy sets: Fuzziness and lack of knowledge. Information Sciences 228, 61-74 (2013)

[17] J.T. Rodríguez, C. Franco, J. Montero. On the semantics of bipolarity and fuzziness. Advances in Intelligent and Soft Computing 107, 193-205 (2012)

[18] B. Roy. The outranking approach and the foundations of ELECTRE methods. Theory and Decision 31, 49-73 (1991)

[19] M. Sayadi, M. Heydari, K. Shahanaghi. Extension of VIKOR method for decision making problem with interval numbers. Applied Methematical Modelling 33, 2257-2262 (2009)

[20] R. Yager, F. Petry. A framework for linguistic relevance feedback in content-based image retrieval using fuzzy logic. Information Sciences 173, 337-352 (2005)

[21] L. Zadeh. The concept of a linguistic variable and its application to approximate reasoning-I. Information Sciences 8, 199-249 (1975)