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Minimum cost connection networks: truth-telling and implementation*

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Abstract

In the present paper we consider the allocation of cost in connection networks. Agents have connection demands in form of pairs of locations they want to be connected. Connections between locations are costly to build. The problem is to allocate costs of networks satisfying all connection demands. We use three axioms to characterize allocation rules that truthfully implement cost minimizing networks satisfying all connection demands in a game where: (1) a central planner announces an allocation rule and a cost estimation rule; (2) every agent reports her own connection demand as well as all connection costs; and, (3) the central planner selects a cost minimizing network satisfying reported connection demands based on *estimated* connection costs and allocates *true* connection costs of the selected network.

Keywords: Axiomatic characterization, Connection networks, Cost sharing, Implementation, Truth-telling.

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1 Introduction

Motivation and overview of the paper: In the present paper we consider cost allocation in the connection network (CN) model used in Anshelevich et al. (2008), Chen et al. (2010), Juarez & Kumar (2013) and Moulin (2013). Agents have connection demands in form of pairs of locations they want to be directly or indirectly connected. Connections between locations are costly to establish. Several agents can use the same connection as part of paths satisfying their connection demands. Therefore connections are public goods.

The German Hansa is an illustrative example of network building in practice. The Hansa started in the middle of the 12th century as an association of north German merchants and developed into a community of cities in the middle of the 14th century. The Hansa aimed at obtaining trading privileges for its merchants in northern Europe as well as protecting and supporting merchants with Hanseatic privileges trading in northern Europe. The Hansa is an example of a network with locations being towns whose merchants had Hanseatic privileges and their markets and connections being roads between towns and markets.

In the middle of the 12th century north German merchants regularly visiting or permanently settling in Gotland formed a community. There were commercial possibilities in Novgorod as well as Polotsk, Vitebsk and Smolensk on the river Dvina in Russia. However the pagan habitants of Finland and the Baltic countries made trade very risky. Around 1200 bishop Albert led a crusade into the Baltic countries. The Gotland community contributed to the crusade by equipping hundreds of crusaders and providing transportation. Lübeck supported the crusade. In 1241 Hamburg and Lübeck agreed to share the cost of keeping the roads between the two towns free from brigands. Both the crusade and the fight against brigandage can be seen as establishing new connections and making these connections safer can be seen as a public good for merchants with Hanseatic privileges and everybody else.

A minimum cost connection network (MCCN) is a network minimizing total costs subject to the constraint that all connection demands have to be satisfied. An allocation rule maps MCCNs, connection demands and connection costs to cost shares to all agents. Depending on the allocation rule there can be a potential conflict between overall welfare aimed at minimizing total cost and individual welfare aimed at minimizing individual cost shares. We characterize the set of allocation rules that truthfully implement MCCNs.

In the minimum cost spanning tree (MCST) model one location is a source and every other location is inhabited by an agent. Every agent wants to be connected to the source making the MCST model a special case of the CN model. Indeed undemanded locations (Steiner nodes) and arbitrary connection demands are allowed in the CN model. In Hougaard & Tvede (2012) we characterized allocation rules that truthfully implement MCCNs in the MCST model in a three stage game:

(1) A central planner announces a cost allocation rule and a cost estimation rule.

(2) Every agent reports all connection costs.

(3) The central planner selects a MCCN based on *estimated* connection costs and allocates *true* costs of the selected network.

An allocation rule is reductionist provided it depends on irreducible costs, where the irreducible cost of a connection is: in case the connection is used, the true cost; and, in case the connection is unused, the lowest cost for which it needs not be used. An allocation rule is monotonic provided it is monotonic in irreducible costs. In Hougaard & Tvede (2012) we showed that an allocation rule implements MCCNs truthfully if and only if it is reductionist and monotonic for the MCST model.

The set of reductionist and monotonic rules is quite large and includes fixed relative cost shares rules such as the equal split rule, the folk rule and the rest of the family of obligation rules introduced and analyzed in Tijs et al. (2006). An important notion is Stand Alone core stability: no group of agents should pay more than the minimum cost for a network satisfying their connection demands. Fixed relative cost shares rules are not Stand Alone core stable. However the family of obligation rules are Stand Alone core stable. Therefore reductionism and monotonicity are compatible with Stand Alone core stability. Hence voluntary participation – even in the strong form of Stand Alone core stability – and truthful implemention of MCCNs are compatible in the MCST model.

Our characterization of allocation rules that truthfully implements MCCNs in the MCST model does not generalize to the CN model. Indeed irreducible costs need not be defined in case of undemanded locations. As a consequence both reductionism and monotonicity need not be defined in case of undemanded locations. Therefore we focus of two crucial properties of reductionism and monotonicity and study the set of allocation rules having these two properties.

An implication of reductionism is that connection costs of unused connections do not influence cost shares. This property is denoted *Independence of Unobserved Information* (IUI). An implication of monotonicity is that cost shares are independent of the selected MCCN in case of multiple MCCNs. This property is denoted *Network Independence* (NI). The property that cost shares are homogeneous of degree one in connection costs is denoted *Scale Invariance*. For the CN model with undemanded locations we characterize allocation rules satisfying IUI and NI as well as SI, IUI and NI (Theorem 1 and Corollary 1).

In order to consider truthful implementation of MCCNs in the CN model we modify the game in in Hougaard & Tvede (2012) to include reports of connection demands:

(1) A central planner announces a cost allocation rule and a cost estimation rule.

(2) Every agent reports her connection demand and all connection costs.

(3) The central planner selects a MCCN based on reported demands and *estimated* connection costs and allocates *true* costs of the selected network.

For the CN model with undemanded locations: we show that if an allocation rule truthfully implement MCCNs, then it satisfies IUI and NI (Observations 1 and 2); and, we characterize allocation rules satisfying SI, IUI and NI and truthfully implementing MCCNs (Theorem 2 in case connection demands are private information and Corollary 2 in case connection demands are public knowledge).

In Theorem 1 we show that an allocation rule satisfies IUI and NI if and only if cost shares depend on connection demands, total costs and nothing else. In Corollary 1 we show that an allocation rule satisfies SI, IUI and NI if and only if *relative* cost shares depend on connection demands and nothing else. Therefore for fixed connection demands the set of allocation rules that satisfy SI, IUI and NI and the set of fixed relative cost shares rules are identical. An important notion is Individual Rationality: no agent should pay more than the minimum cost for a path satisfying her connection demands. Clearly Individual Rationality is much weaker than Stand Alone core stability. Theorem 1 implies reductionism and monotonicity, interpreted as IUI and NI, are incompatible with Individual Rationality. Hence voluntary participation – even in the weak form of Individual Rationality – and truthful implemention of MCCNs are incompatible in the CN model.

The German Hansa lived with the tension between Individual Rationality and truthful implementation. In 1284 the Norwegian king restricted the Hanseatic privileges. The German Hansa responded with a blockade. Bremen did not participate in the blockade probably because being part of the blockade would have favoured merchants from other places at least in the short run. Bremen was excluded from the Hansa. The behaviour of Bremen can be interpreted as an attempt to free ride: if the blockade failed, Bremen could continue to trade with Norway; and, if the blockade succeeded, Bremen could benefit from the improved Hanseatic privileges. The Hansa and Denmark were at war from 1367 to 1369 when Denmark asked for peace. The Westphalian towns including Cologne traded mainly with England and the low countries. These towns did not contribute or support the war, but they were not excluded from the Hansa. The behaviour of the Westphalian towns can be interpreted as a reflection of their commercial interests or equivalently connection demands.

The German Hansa could sanction participants by use of fines, confiscation and exclusion, but as mentioned participation was voluntarily. Based on the different responses to Bremen and the Westphalian towns it appears the Hansa dealt with the incompatibility of voluntary participation and truthful implementation by accepting that cost shares should depend on commercial interests or equivalently connection demands. Probably it worked to relate cost shares and commercial interests because commercial interests of every merchant were known by other merchants. The parts on the Hansa closely follow Dollinger (1970).

Related literature: It is shown in Megiddo (1978) and Tamir (1991) that the set of Stand Alone core stable allocations can be empty for the CN model in case of undemanded locations. Therefore it is trivial that reductionism and monotonicity are not compatible with Stand Alone core stability. However it is less trivial that reductionism and monotonicity are incompatible with Individual Rationality.

In Moulin (2013) the folk rule is extended from the minimum cost spanning tree model to two subclasses of the CN model for which the set of Stand Alone core stable allocation rules is nonempty.

Implementation in the CN model has been analyzed in Anshelevich et al. (2008), Chen et al. (2010) and Juarez & Kumar (2013). In all three papers the same game is considered:

(1) A central planner announces a cost allocation rule.

(2) Every agent reports a path between the pair of locations she wants to be connected.

(3) The central planner selects the network consisting of all reported paths and allocates costs of the selected network.

Compared with our game every agent has less impact on the selection of network, because the other agents reports their own paths and these paths are part of the selected network.

In the two former papers properties of a specific allocation rule (the cost of every connection is divided equally between agents whose reported paths include the connection) are studied. It is shown that: Nash equilibria exist; all Nash equilibria can be inefficient; and, there are bounds on the ratios between the cost of the cheapest Nash equilibrium and the cost of a MCCN (price of stability) and the cost of the most expensive Nash equilibrium and the cost of a MCCN (price of anarchy). In the latter paper attention is restricted to the set of allocation rules that depends on costs of reported paths, total costs and nothing else. It is shown that the set of allocation rules implementing MCCNs and the set of fixed relative cost shares rules are identical. Compared with the present paper on the one hand a much smaller set of allocation rules is considered and on the other hand every agent has less impact on the network selection. Since attention is restricted to allocation rules that do not depend on connection demands, our Corollary 1 implies that the only allocation rules satisfying SI, IUI and NI are fixed relative cost shares rules.

Implementation in the MCST model has been analyzed in Bergantinos & Lorenzo (2004, 2005), Bergantinos & Vidal-Puga (2010) and Hougaard & Tvede (2012). All four papers consider existence of and properties of Nash equilibria.

Plan of the paper: In Section 2 we introduce the set up and the axioms; in Section 3 we characterize allocation rules satisfying our axioms; in Section 4 we study implementation of MCCNs; in Section 5 we discuss some of generalizations of our set up; and finally, in Section 6 we end with a few final remarks.

2 The CN model

In the present section we introduce the set up as well as the axioms.

Set up

Let $\mathscr{M} = \{1, \ldots, m\}$ be a set of finitely many agents and $\mathscr{N} = \{1, \ldots, n\}$ a set of finitely many locations. The set of connections between pairs of locations is $\mathscr{N}^2 = \mathscr{N} \times \mathscr{N}$. Every agent $i \in \mathscr{M}$ has a connection demand in the sense that she wants a pair of locations $(a_i, b_i) \in \mathscr{N}^2$ to be connected. A *connection structure* P is a collection of connection demands $(a_i, b_i)_{i \in \mathscr{M}}$. Let $\mathscr{P}$ be the set of connection structures. A *cost structure* C describes the cost of connecting any pair of locations and is defined by a map $c : \mathscr{N}^2 \to \mathbb{R}_+$ with

- $c_{jj} = 0$ for all j.
- $c_{jk} > 0$ for all j and k with $j \neq k$.
- $c_{kj} = c_{jk}$ for all j and k.

A *connection problem* is a connection structure and a cost structure (P, C).

A *connection network* is a graph g such that for every agent i there is a path $p_i(g) = a_i \ldots b_i$ between a_i and b_i in g and $g = \cup_i p_i(g)$. Let $v(g, C)$ be the total cost of a connection network

$$v(g, C) \;=\; \sum_{jk \in g} c_{jk}.$$

A *Minimum Cost Connection Network* (MCCN) is a connection network g with $v(g, C) \leq v(h, C)$ for every connection network h. The set of MCCNs is non-empty and finite because the set of connection networks is non-empty and finite. Let $\mathscr{MCCN}(P, C)$ be the set of MCCNs for a given connection problem (P, C). Clearly every MCCN is either a tree or a forest (a collection of trees) because if there is a cycle in a connection network, then removing any connection in the cycle does not change whether connection demand satisfied or not.

A *cost allocation problem* (g, P, C) is a connection network g and a connection problem (P, C) such that $g \in \mathscr{MCCN}(P, C)$. Let $\mathscr{X}$ be the set of cost allocation problems (g, P, C) where at least one location is not part of the connection demand of any agent $\mathscr{N} \setminus \cup_i \{a_i, b_i\} \neq \emptyset$.

Allocation rules and axioms

For a given cost allocation problem (g, P, C) the total cost of the MCCN g has to be shared among agents in $\mathscr{M}$. An *allocation rule* $\phi : \mathscr{X} \to \mathbb{R}^m_+$ maps a cost allocation problem to an

m-dimensional vector of positive cost shares,

$$\phi(g,P,C) \;=\; (\phi_1(g,P,C),\ldots,\phi_m(g,P,C)),$$

such that budget-balance is obtained, so $\sum_{i\in\mathscr{M}}\phi_i(g,P,C) = v(g,C)$.

Most game theoretic solution concepts such as the Shapley value map connection problems (P,C) rather than cost allocation problems (g,P,C) to cost shares. However maps ϕ from cost allocation problems to cost shares can be used to define maps Φ from connection problems to cost shares. Indeed let Φ be defined by

$$\Phi(P,C) \;=\; \frac{1}{|\mathscr{MCCN}(P,C)|}\sum_{g\in\mathscr{MCCN}(P,C)}\phi(g,P,C).$$

We use maps from cost allocation rather than connection problems because the properties of allocation rules we consider become weaker and clearer.

The first property is standard.

Scale Invariance (SI) *For all $(g,P,C)\in\mathscr{X}$ and $\lambda>0$, $\phi(g,P,\lambda C)=\lambda\phi(g,P,C)$.*

SI states that if all connection costs are increased by the same factor, then all cost shares should increase by that factor. The last two properties are at the heart of our characterization.

Independence of Unobserved Information (IUI) *For all $(g,P,C),(g,P,D)\in\mathscr{X}$ with $c_{jk}=d_{jk}$ for all $jk\in g$, $\phi(g,P,C)=\phi(g,P,D)$.*

IUI states that for two cost allocation problems with identical MCCNs, connection structures and observed connection costs, the allocations of costs should be identical. Hence IUI implies the allocation of cost is independent of costs of unobserved connections.

Network Invariance (NI) *For all $(g,P,C),(h,P,C)\in\mathscr{X}$, $\phi(g,P,C)=\phi(h,P,C)$.*

NI states that for a connection problem with multiple MCCNs, the allocations of costs should be identical for all MCCNs. Hence NI implies the allocation of costs is independent of the chosen MCCN.

In case of constructing a network, depending on the perception of fairness, costs of unused connections can be allowed to have more or less influence on the allocation of total cost of the realized network. One position is that since the cost of a realized connection network is observed and costs of unused connections are unobserved estimates, there is no obvious reason for letting the allocation of realized cost depend on the costs of unused connections. Moreover for allocation rules that allow costs of unused connections to influence the allocation of total cost there is a conflict between agents over cost estimates of unused connections, This conflict can be an obstacle for implementation of the efficient network as we demonstrate below. All in all, IUI can be seen to reflect a positive as well as a normative approach to cost sharing. Allocation rules such as the cost sharing protocol considered in Chen et al. (2010) for which the cost of every connection in the realized graph is split equally between its users satisfy IUI. Allocation rules violating IUI include most game theoretic solution concepts.

In connection problems with several MCCNs total costs are identical for all MCCNs. Therefore there is no reason for choosing one MCCN over another. Hence it is natural that the allocation of costs should not depend on the chosen MCCN. Moreover for allocation rules that allow the choice of MCCN to influence the allocation of cost there is a conflict between agents over which MCCN to choose. This conflict can be an obstacle for implementation of the efficient network as we demonstrate below. All in all, NI can be seen to reflect a positive as well as a normative approach to cost sharing. Most game theoretic solution concepts satisfy NI. Allocation rules violating NI include the cost sharing protocol considered in Chen et al. (2010).

3 Characterization results

In the present section we characterize allocation rules satisfying IUI and NI as well as allocation rules satisfying SI, IUI and NI.

Simple allocation rules

We consider allocation rules for which cost shares depend on the connection structure P and the total cost of the MCCN $v(g,C)$ and no other feature of the cost allocation problem (g,P,C).

Definition 1 *A* **simple allocation rule** *is an allocation rule $\phi : \mathscr{X} \to \mathbb{R}^m_+$ for which there is a map $\Gamma : \mathscr{P} \times \mathbb{R}_{++} \to \mathbb{R}^m_+$ such that for all $(g,P,C) \in \mathscr{X}$,*

$$\phi(g,P,C) \;=\; \Gamma(P, v(g,C)).$$

For simple allocation rules, the allocation of costs depends on the connection structure as well as the total cost, but not on any other property of the cost structure.

Theorem 1 *An allocation rule ϕ satisfies IUI and NI if and only if it is simple.*

In Hougaard & Tvede (2012) we consider the minimum cost spanning tree model where $\mathscr{M} = \{1,\ldots,m\}$, $\mathscr{N} = \{0,\ldots,m\}$ and $(a_i,b_i) = (0,i)$ for all i. Moreover we show that the set of allocation rules satisfying SI, IUI and NI is quite large and includes the equal split rule and the family of obligation rules. The family of obligation rules are Stand Alone core stable. Theorem 1 implies that adding locations not inhabited by any agent to the minimum cost spanning tree model shrinks the set of allocation rules dramatically. Indeed the set of allocation rules satisfying IUI and NI contains no Individual Rational allocation rule.

Proof of Theorem 1

We leave it to the reader to check that simple allocation rules satisfy IUI and NI. Consequently we focus on the converse claim. The proof consists of four parts.

First, we make the following preliminary observation. Consider a cost allocation problem (g,P,C) and a finite number of pairs of MCCNs and cost structures $(g^1,C^1),\ldots,(g^N,C^N)$ such that

$$g,g^1 \in \mathscr{MCCN}(P,C^1) \text{ and } c^1_{jk} = c_{jk} \text{ for all } jk \in g.$$

$$g^1,g^2 \in \mathscr{MCCN}(P,C^2) \text{ and } c^2_{jk} = c^1_{jk} \text{ for all } jk \in g^1.$$

$$\vdots$$

$$g^{N-1},g^N \in \mathscr{MCCN}(P,C^N) \text{ and } c^N_{jk} = c^{N-1}_{jk} \text{ for all } jk \in g^{N-1}.$$

Then $\phi(g^N,P,C^N) = \phi(g,P,C)$ according to IUI and NI.

Part 1: (A Cost Moving Lemma)

Lemma 1 *Consider a cost allocation problem (g,P,C). Suppose that there is a location u with $u \in \mathscr{N}$ and $u \notin g$. Then for every pair of connections rs and st in g and all cost structures C' with $c'_{rs} + c'_{st} = c_{rs} + c_{st}$ and $c'_{jk} = c_{jk}$ for all other connections, $\phi(g,P,C) = \phi(g,P,C')$.*

Proof: Without loss of generality assume that $c'_{rs} < c_{rs}$ and $c'_{st} > c_{st}$.

In case removing rs and st from g and adding rt to g result in a graph, which is a connection network, consider the following three steps.

Step 1: Define g^1 by $g^1 = g$ and define C^1 by

$$c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g \\ \max\{c_{jk}, v(g,C)\} & \text{for all other connections.} \end{cases}$$

Then $g, g^1 \in \mathscr{MCCN}(P,C^1)$ and $c^1_{jk} = c_{jk}$ for all $jk \in g$. Therefore $\phi(g^1,P,C^1) = \phi(g,P,C)$ according to IUI.

Step 2: Define g^2 by removing rs and st from g^1 and adding rt. Then g^2 is a connection network. Define C^2 by

$$c^2_{jk} = \begin{cases} c^1_{rs} + c_{st} & \text{for } jk = rt \\ c^1_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^1, g^2 \in \mathscr{MCCN}(P,C^2)$ and $c^2_{jk} = c^1_{jk}$ for all $jk \in g^1$. Therefore $\phi(g^2,P,C^2) = \phi(g^1,P,C^1)$ according to IUI and NI.

Step 3: Define g^3 by removing rt from g^2 and adding rs and st to g^2. Then g^3 is a connection network. Indeed $g^3 = g^1$. Define C^3 by

$$c^3_{jk} = \begin{cases} c'_{rs} & \text{for } jk = rs \\ c'_{st} & \text{for } jk = st \\ c^2_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^2, g^3 \in \mathscr{MCCN}(P,C^3)$ and $c^3_{jk} = c^2_{jk}$ for all $jk \in g^1$. Therefore $\phi(g^3,P,C^3) = \phi(g^2,P,C^2)$ according to IUI and NI. Moreover $c^3_{rs} = c'_{rs}$ and $c^3_{st} = c'_{st}$.

In case removing rs and st from g and adding rt to g result in a graph, which is not a connection network, consider the following four steps.

Step 1: Define g^1 by $g^1 = g$ and define C^1 by

$$c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g \\ \max\{c_{jk}, v(g,C)\} & \text{for all other connections.} \end{cases}$$

Then $g, g^1 \in \mathscr{MCCN}(P,C^1)$ and $c^1_{jk} = c_{jk}$ for all $jk \in g$. Therefore $\phi(g^1,P,C^1) = \phi(g,P,C)$ according to IUI.

Step 2: Define g^2 by removing rs and st from g^1 and adding ru, su and tu to g^1. Then g^2 is a connection network. Define C^2 by

$$c^2_{jk} = \begin{cases} \max\{c'_{rs}, c^1_{rs} - c^1_{st}\} & \text{for } jk = ru \\ c^1_{rs} - \max\{c'_{rs}, c^1_{rs} - c^1_{st}\} & \text{for } jk = su \\ c^1_{st} & \text{for } jk = tu \\ c^1_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^1, g^2 \in \mathscr{MCCN}(P, C^2)$ and $c^2_{jk} = c^1_{jk}$ for all $jk \in g^1$. Therefore $\phi(g^2, P, C^2) = \phi(g^1, P, C^1)$ according to IUI and NI.

Step 3: Define g^3 by removing ru, su and tu from g^2 and adding rs and st to g^2. Then g^3 is a connection network. Indeed $g^3 = g^1$. Define C^3 by

$$c^3_{jk} = \begin{cases} c^2_{ru} & \text{for } jk = rs \\ c^2_{su} + c^2_{tu} & \text{for } jk = st \\ c^2_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^2, g^3 \in \mathscr{MCCN}(P, C^3)$ and $c^3_{jk} = c^2_{jk}$ for all $jk \in g^1$. Therefore $\phi(g^3, P, C^3) = \phi(g^2, P, C^2)$ according to IUI and NI. Moreover $c^3_{rs} < c^1_{rs}$ and $c^3_{st} > c^1_{st}$.

Step 4: Repeat steps 2 and 3 until the $c^3_{rs} = c'_{rs}$ and $c^3_{st} = c'_{st}$. □

Part 2: (Another Cost Moving Lemma)

Lemma 2 *Consider a cost allocation problem (g, P, C). For every pair of connections rs and $r's'$ in g, where there is no path between r and r' in g, and all cost structures C' with $c'_{rs} + c'_{r's'} = c_{rs} + c_{r's'}$ and $c'_{jk} = c_{jk}$ for all other connections, $\phi(g, P, C) = \phi(g, P, C')$.*

Proof: Without loss of generality assume that $c'_{rs} < c_{rs}$ and $c'_{r's'} > c_{r's'}$.

Step 1: Define g^1 by $g^1 = g$ and define C^1 by

$$c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g \\ \max\{c_{jk}, v(g, C)\} & \text{for all other connections.} \end{cases}$$

Then $g, g^1 \in \mathscr{MCCN}(P, C^1)$ and $c^1_{jk} = c_{jk}$ for all $jk \in g$. Therefore $\phi(g^1, P, C^1) = \phi(g, P, C)$ according to IUI.

Step 2: Define g^2 by removing rs and $r's'$ from g^1 and adding rr', $r's$ and ss' to g^1. Then g^2 is a connection network. Define C^2 by

$$c^2_{jk} = \begin{cases} \max\{c'_{rs}, \frac{c^1_{rs}}{2}\} & \text{for } jk = rr' \\ c^1_{rs} - \max\{c'_{rs}, \frac{c^1_{rs}}{2}\} & \text{for } jk = r's \\ c_{r's'} & \text{for } jk = ss' \\ c^1_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^1, g^2 \in \mathscr{MCCN}(P, C^2)$ and $c^2_{jk} = c^1_{jk}$ for all $jk \in g^1$. Therefore $\phi(g^2, P, C^2) = \phi(g^1, P, C^1)$ according to IUI and NI.

Step 3: Defined g^3 by removing rr', $r's$ and ss' from g^2 and adding rs and $r's'$ to g^2, so $g^3 = g$. Define C^3 by

$$c^3_{jk} = \begin{cases} c^2_{rr'} & \text{for } jk = rs \\ c^2_{r's} + c_{ss'} & \text{for } jk = r's' \\ c^2_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^2, g^3 \in \mathscr{MCCN}(P, C^3)$ and $c^3_{jk} = c^2_{jk}$ for all $jk \in g^1$. Therefore $\phi(g^3, P, C^3) = \phi(g^2, P, C^2)$ according to IUI and NI.

Step 4: Repeat steps 2 and 3 until the $c^3_{rs} = c'_{rs}$ and $c^3_{r's'} = c'_{r's'}$. □

Part 3: (A Lemma showing independence of connection network and cost structure)

Lemma 3 *Consider two cost allocation problem (g, P, C) and (h, P, D) with $v(g, C) = v(h, D)$. Then $\phi(g, P, C) = \phi(h, P, D)$.*

Proof: First the cost allocation problem (g, P, C) is transformed into (g^2, P, C^2) with $\phi(g^2, P, C^2) = \phi(g, P, C)$ such that there is a location u with $u \notin g^2$. Consider the following two steps.

Step 1: Define g^1 by $g^1 = g$ and define C^1 by

$$c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g \\ v(g, C) & \text{for all other connections.} \end{cases}$$

Then $g, g^1 \in \mathscr{MCCN}(P, C^1)$ and $c^1_{jk} = c_{jk}$ for all $jk \in g$. Therefore $\phi(g^1, P, C^1) = \phi(g, P, C)$ according to IUI.

Step 2: Define g^2 by removing ru and su from g^1 and adding rs to g^1 as well as replacing all other connections to u in g^1 with connections to r. Then g^2 is a connection network. Define C^2 by

$$c^2_{jk} = \begin{cases} c^1_{ru} + c^1_{su} & \text{for } jk = rs \\ c^1_{ju} & \text{for all } j \text{ with } ju \in g \text{ and } k = r \\ c^1_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^1, g^2 \in \mathscr{MCCN}(P, C^2)$ and $c^2_{jk} = c^1_{jk}$ for all $jk \in g^1$. Therefore $\phi(g^2, P, C^2) = \phi(g^1, P, C^1)$ according to IUI and NI.

Let M be a subset of $\mathscr{M}$ such that

- If h is a connection network for $(a_i, b_i)_{i\in M}$, then h is a connection network for P.
- For all $i \in M$ there exists a connection network h for $(a_{i'}, b_{i'})_{i'\in M\setminus\{i\}}$ such that h is not a connection network for P.

Second (g^2, P, C^2) is transformed into $(\tilde{g}, P, \tilde{C})$ with $\phi(\tilde{g}, P, \tilde{C}) = \phi(g^2, P, C^2)$ such that $\tilde{g} = \cup_{i\in M}\{a_i, b_i\}$ and $\tilde{C}$ is arbitrary with $\sum_{i\in M} \tilde{c}_{a_i b_i} = v(g, P, C)$ and $\tilde{c}_{jk} = v(g, P, C)$ for all other connections. Consider the following three steps.

Step 3: Pick $i \in M$ with $a_i b_i \notin g^2$. Then there exists a connection $j'k'$ in the path $p_i(g^2)$ between a_i and b_i such that $j'k' \notin \{(a_{i'}, b_{i'})_{i'\in M}\}$. Apply Lemma 1 to move connection costs in the path $p_i(g^2)$ such that $c_{j'k'} \geq \sum_{jk\in p_i(g^2)\setminus j'k'} c_{jk}$. Define h by removing $j'k'$ from g^2 and adding $a_i b_i$ to g^2. Define g^3 by removing all connections in $h \setminus \cup_{i'\in M} p_{i'}(h)$ from h. Define C^3 by

$$c^3_{jk} = \begin{cases} c^2_{j'k'} + \sum_{j''k''\in h\setminus\cup_{i'\in M} p_{i'}(h)} c^2_{j''k''} & \text{for } jk = a_i b_i \\ c^2_{jk} & \text{for all other connections.} \end{cases}$$

Then $g^2, g^3 \in \mathscr{MCCN}(P, C^3)$ and $c^3_{jk} = c^2_{jk}$ for all $jk \in g^2$. Therefore $\phi(g^3, P, C^3) = \phi(g^2, P, C^2)$ according to IUI and NI.

Step 4: Repeat step 3 until $a_i b_i \in g^3$ for all $i \in M$ so $g^3 = \cup_{i\in M}\{a_i, b_i\}$.

Step 5: Apply Lemmas 1 and 2 to move cost to an arbitrary $\tilde{C}$ with $\sum_{i\in M} \tilde{c}_{a_i b_i} = v(g, P, C)$ and $\tilde{c}_{jk} = v(g, P, C)$ for all other connections. Then $\tilde{g} \in \mathscr{MCCN}(P, \tilde{C})$ for $\tilde{g} = g^3$ and $\phi(\tilde{g}, P, \tilde{C}) = \phi(g^3, P, C^3)$.

All in all $\phi(g, P, C) = \phi(\tilde{g}, P, \tilde{C}) = \phi(h, P, D)$. □

Part 4: (Summing up)

According to Lemma 3 the allocation of cost depends on the connection structure P and the total cost of the MCCN $v(g, C)$ and no other feature of the cost allocation problem (g, P, C). Therefore the allocation rule is simple.

Super simple allocation rules

We consider simple allocation rules that are linear in total cost. For simple allocation rules, the allocation of cost can depend on total cost in quite weird ways. Indeed, consider the following example: If total cost is a rational number $v(g, C) \in \mathbb{Q}$, costs are allocated such that

$$\phi(g, P, C) = \frac{v(g, C)}{m}$$

for all i and if the total cost is an irrational number $v(g,C) \in \mathbb{R}\backslash\mathbb{Q}$, costs are allocated such that

$$\phi_i(g,P,C) \;=\; \frac{w_i}{\sum_{j\in\mathscr{M}} w_j} v(g,C)$$

where

$$w_i = \sum_{j\in\{a_i,b_i\}} |\{i' \mid j = a_{i'}\}| + |\{i' \mid j = b_{i'}\}|$$

for all i. This rule obviously violates scale invariance SI although each of the two separate parts are ‘well behaved’ considered by themselves. The first part of the cost allocation rule is the equal split rule where cost shares of all agents are identical. The second part is a rule where the cost share of agent i depends on the number of agents who want to have the same locations as agent i connected. Both parts of the cost allocation rule are examples of simple cost allocation rules which are linear in total cost and therefore satisfies SI.

Therefore we introduce the following class of allocation rules.

Definition 2 *A* **super simple allocation rule** *is an allocation rule $\phi : \mathscr{X} \to \mathbb{R}^m_+$ for which there is a map $\Lambda : \mathscr{P} \to \mathbb{R}^m_+$ such that for all $(g,P,C) \in \mathscr{X}$*

$$\phi(g,P,C) \;=\; \Lambda(P)v(g,C).$$

For super simple allocation rules, the allocation of costs is simple and depends linearly on total cost. Consequently super simple rules are continuous in cost structures.

Corollary 1 *An allocation rule ϕ satisfies SI, IUI and NI if and only if it is super simple.*

Proof: According to Theorem 1 the class of allocation rules satisfying IUI and NI and the class of simple allocation rules are identical

$$\phi(g,P,C) \;=\; \Gamma(P,v(g,C))$$

Total cost is linear in the cost structure

$$v(g,\lambda C) \;=\; \lambda v(g,C).$$

Therefore if a simple allocation rule satisfies SI, then

$$\begin{aligned}
\lambda\Gamma(P,v(g,C)) &= \lambda\phi(g,P,C) \\
&= \phi(g,P,\lambda C) \\
&= \Gamma(P,v(g,\lambda C)) \\
&= \Gamma(P,\lambda v(g,C)).
\end{aligned}$$

Hence the allocation rule is super simple because it is linear in total cost. □

4 Implementation

In the present section we consider a network formation game with the m agents and a central planner. The agents have private information about their connection demands and know all connection costs. The planner is ignorant. The agents need to ensure that their individual connection demands are satisfied and want to minimize their individual cost shares. The planner wants minimize total costs subject to the constraint that all connection demands has to be satisfied.

The game

The network formation game has three stages:

(1) A central planner announces a cost allocation rule and a cost estimation rule.

(2) Every agent reports her connection demands and all connection costs.

(3) The central planner selects a MCCN based on reported demands and *estimated* connection costs and allocates *true* costs of the selected network.

The reports of the agents are used by the planner to estimate the connection and cost structures. Agents can misreport both their individual connection demands (a_i, b_i) and the cost structure C. Misreporting influences the estimates of connection demands and costs. Estimates of connection demands and costs influences the selection of an MCCN. However, the true cost rather than the estimated cost of the selected MCCN is observed and allocated. Therefore misreporting has an indirect rather than direct influence on cost shares.

Formally, the rules of the game consist of an allocation rule $\phi : \mathscr{X} \to \mathbb{R}^m_+$ and a connection cost estimation rule $\tau : \mathbb{R}^m_{++} \to \mathbb{R}_{++}$. Let $\sigma = (\sigma_1, \ldots, \sigma_m)$, where $\sigma_i = (\sigma_i^{jk})_{1 \le j < k \le n}$ for all i and $\sigma_i^{jk} > 0$ for all i and jk, be a collection of individual cost reports. The estimated cost of a connection is supposed to be between the minimum and maximum reported cost

$$\tau(\sigma_1^{jk}, \ldots, \sigma_m^{jk}) \in [\min_i\{\sigma_i^{jk}\}, \max_i\{\sigma_i^{jk}\}]$$

and to be upward unbounded

$$\lim_{\sigma_i^{jk} \to \infty} \tau(\sigma_1^{jk}, \ldots, \sigma_m^{jk}) = \infty$$

for every i. Upward unboundedness implies that every agent is able to influence the cost estimation of the planner.

Let $\omega \in (\mathscr{N} \times \mathscr{N})^m$, where $\omega = (\omega_1, \ldots, \omega_m)$ and $\omega_i = (\alpha_i, \beta_i)$ for all i, be a collection of individual reports of connection demands. For the collection of individual connection demand reports ω the estimated connection structure is $P^e(\omega) = \omega$. For a collection of individual cost reports σ the estimated cost structure is $C^e(\sigma)$, where $c^e_{jk} =$

$\tau(\sigma_i^{jk},\ldots,\sigma_m^{jk})$ for all jk. The planner allocates observed costs. Therefore for every network g in $\mathscr{MCCN}(P^e(\omega),C^e(\sigma))$ the allocation of observed costs is $\phi(g,P^e(\omega),C)$. The planner randomly selects a network g in $\mathscr{MCCN}(P^e(\omega),C^e(\sigma))$. Hence for fixed collections of individual reports (ω,σ) the expected allocation of cost is

$$\Phi(P^e(\omega),C^e(\sigma),C) \;=\; \frac{1}{|\mathscr{MCCN}(P^e(\omega),C^e(\sigma))|}\sum_{g\in\mathscr{MCCN}(P^e(\omega),C^e(\sigma))}\phi(g,P^e(\omega),C).$$

Agents can choose their reports strategically.

Equilibrium

The notion of equilibrium is Nash equilibrium.

Definition 3 *A* **Nash equilibrium** *is a collection of individual reports $(\bar{\omega},\bar{\sigma})$ such that for every agent i and all reports (ω_i,σ_i),*

$$\Phi_i(P^e(\omega_i,\bar{\omega}_{-i}),C^e(\sigma_i,\bar{\sigma}_{-i}),C) \;\geq\; \Phi_i(P^e(\bar{\omega}),C^e(\bar{\sigma}),C).$$

No truth-telling without IUI and NI

In the two observations below we shown that if truth-telling is a Nash equilibrium, then the allocation rule satisfies both IUI and NI provided the estimation rule is well behaved.

Misreporting connection cost of unused connections influences the cost estimates of unused connections. Moreover if an allocation rule does not satisfy IUI, cost estimates of unused connections influence cost shares. Therefore agents can manipulate their cost shares by misreporting provided IUI is not satisfied. Furthermore misreporting need not be revealed.

Observation 1 *Suppose an allocation rule ϕ is continuous in cost structures and an estimation rule τ is continuous and unbounded in the sense that for every i,*

$$\lim_{\sigma_i^{jk}\to 0}\tau(\sigma_1^{jk},\ldots,\sigma_m^{jk}) \;=\; 0 \textit{ and } \lim_{\sigma_i^{jk}\to\infty}\tau(\sigma_1^{jk},\ldots,\sigma_m^{jk}) \;=\; \infty.$$

If truth-telling is a Nash equilibrium, then ϕ satisfies IUI.

Proof: Suppose a cost allocation rule ϕ does not satisfy IUI. Then there is a pair cost allocation problems (g,P,C) and (g,P,D) with $c_{jk}=d_{jk}$ for all $jk\in g$ and an agent i such that $\phi_i(g,P,C)\neq\phi_i(g,P,D)$. For $\varepsilon>0$ define two other cost structures C^1 and D^1 by

$$c^1_{jk} \;=\; \begin{cases} c_{jk} & \text{for } jk\in g \\ c_{jk}+\varepsilon & \text{for all other connections}\end{cases}$$

and

$$d^1_{jk} = \begin{cases} d_{jk} & \text{for } jk \in g \\ d_{jk} + \varepsilon & \text{for all other connections.} \end{cases}$$

Then g is the unique MCCN for connection problems (P, C^1) and (P, D^1) for all ε. For ε sufficiently small $\phi_i(g, P, C^1) \neq \phi_i(g, P, D^1)$ because ϕ is continuous in cost structures. Without loss of generality suppose $\phi_i(g, P, C^1) > \phi_i(g, P, D^1)$. For the connection problem (P, C^1) if all agents except agent i are telling the truth, then the strategy (ω_i, σ_i), where $\omega_i = (a_i, b_i)$ and σ_i is such that $\tau(\sigma_i^{jk}, (c_{jk})_{i' \neq i}) = d^1_{jk}$ lowers the cost share of agent i. There exists σ^i such that $\tau(\sigma_i^{jk}, (c_{jk})_{i' \neq i}) = d^1_{jk}$ because τ is continuous and unbounded. □

Misreporting connection cost influences cost estimates. Moreover cost estimates influence the set of estimated MCCN. If an allocation rule does not satisfy NI, then cost shares depends on the selected MCCN. Therefore agents can manipulate their cost shares by misreporting provided NI is not satisfied.

Observation 2 *Suppose an allocation rule ϕ is continuous in cost structures and an estimation rule τ is continuous and upward unbounded in the sense that for every i,*

$$\lim_{\sigma_i^{jk} \to \infty} \tau(\sigma_1^{jk}, \ldots, \sigma_m^{jk}) = \infty$$

If truth-telling is a Nash equilibrium, then ϕ satisfies NI.

Proof: Suppose a cost allocation rule ϕ does not satisfy NI. Then there is a connection problem (P, C) and an agent i such that

$$\min_{g \in \mathscr{MCCN}(P,C)} \phi_i(g, P, C) < \max_{h \in \mathscr{MCCN}(P,C)} \phi_i(h, P, C)$$

for some i. Let g be defined by $\phi_i(g, P, C) \leq \phi_i(h, P, C)$ for all $h \in \mathscr{MCCN}(P, C)$. For $\varepsilon > 0$ define another cost structure C^1 by

$$c^1_{jk} = \begin{cases} c_{jk} & \text{for } jk \in g \\ c_{jk} + \varepsilon & \text{for all other connections} \end{cases}$$

Then g is the unique MCCN for connection problem (P, C^1) for all ε. For ε sufficiently small

$$\phi_i(g, P, C^1) < \frac{1}{|\mathscr{MCCN}(P, C)|} \sum_{g' \in \mathscr{MCCN}(P,C)} \phi_i(g, P, C)$$

because ϕ is continuous in cost structures. For connection problem (P, C) if all agents except agent i are telling the truth, then the strategy (ω_i, σ_i), where $\omega_i = (a_i, b_i)$ and σ_i is such that $\tau(\sigma_i^{jk}, (c_{jk})_{i' \neq i}) = c^1_{jk}$ lowers the expected cost share of agent i. There exists σ_i such that $\tau(\sigma_i^{jk}, (c_{jk})_{i' \neq i}) = c^1_{jk}$ because τ is continuous and upward unbounded. □

Truth-telling

SI is standard and both IUI and NI are necessary for truthfully reporting as shown in Observations 1 and 2. Consequently we focus on super simple allocation rules.

Let $P_{-i} = (a_{i'}, b_{i'})_{i' \neq i}$ be a collection of connection demands for all agents except agent i. Then a super simple allocation rule is implementable if and only if for all i and P_{-i}, the relative cost share of agent i is independent of her strategy (α_i, β_i).

Theorem 2 *Suppose an allocation rule ϕ is super simple. Then truth-telling is a Nash equilibrium if and only if $\Lambda_i((a_i,b_i),P_{-i}) = \Lambda_i((\alpha_i,\beta_i),P_{-i})$ for all i, P_{-i}, (a_i,b_i) and (α_i,β_i).*

Proof: Suppose $\Lambda_i((a_i,b_i),P_{-i}) < \Lambda_i((\alpha_i,\beta_i),P_{-i})$ for some P_{-i} and (a_i,b_i) and (α_i,β_i). Consider a cost structure C such that for some $i' \neq i$ a path between $a_{i'}$ and $b_{i'}$ going through all locations is the unique MCCN. Suppose the true connection structure is P. Then agent i can lower her cost share by changing her strategy from (a_i,b_i) to (α_i,β_i). Therefore truth-telling is not a Nash equilibrium.

Suppose truth-telling is not a Nash equilibrium for some connection problem (P,C). Then there are i with $(\alpha_i,\beta_i) \neq (a_i,b_i)$ and h such that

$$\Lambda_i((\alpha_i,\beta_i),P_{-i})v(h,C) \; < \; \Lambda_i(((a_i,b_i),P_{-i}))v(g,C).$$

Therefore $\Lambda_i((\alpha_i,\beta_i),P_{-i}) < \Lambda_i((a_i,b_i),P_{-i})$, because $v(h,C) \geq v(g,C)$. □

Theorem 2 implies that the price of stability is one provided the allocation rule satisfies the assumption in the theorem. However a simple example shows that the the price of anarchy is unbounded. Consider a situation with at least two agents $i \in \{1,2\}$ and three locations $j \in \{1,2,3\}$. Suppose the connection demands are $(a_i,b_i) = (1,2)$ for both i and the cost structure is $c_{12} = 1$, $c_{13} = 2(1+k)$ and $c_{23} = k$ where $k > 0$. Then the unique MCCN is $g = 12$ with $v(g,C) = 1$. Let the allocation rule be the equal split rule. Suppose both agents misreport their connection demands $(\alpha_i,\beta_i) = (1,3)$ for both i and report the true cost structure. Based on the reports the network $h = 12,23$ is selected with $v(h,C) = 1+k$. Moreover it is not possible for any of the two agents to bring down her cost share by changing her report. Therefore the price of anarchy is $1+k$ where k is arbitrary.

Suppose connection demands are known, so every agent reports only connection costs rather than both her connection demand and connection costs. Then for arbitrary rules of the game the set of MCCNs for the estimated cost structure is a subset of the set of MCCNs for the true cost structure. Hence the price of anarchy drops to one.

Corollary 2 *Suppose connection demands are known. If σ is a Nash equilibrium, then $\mathscr{MCCN}(P,C^e(\sigma)) \subset \mathscr{MCCN}(P,C)$.*

Proof: The proof follows the proof of Theorem 2 in Hougaard & Tvede (2012). □

5 Generalizations

Lemma 2 shows our characterizations of simple and super simple allocation rules extend to cost allocation problems (g,P,C) with $\mathcal{N} \setminus \cup_i \{a_i,b_i\} = \emptyset$ provided g is a nontrivial forest.

Using the idea of routing-proofness from Moulin (2013) every agent i could misrepresent her connection demand by splitting into m_i aliases who each reports a connection demand. The characterization in Theorem 2 of allocation rules that truthfully implements MCCNs remains valid provided the condition in Theorem 2 is modified to $\Lambda_i((a_i,b_i),P_{-i}) \leq \sum_{k=1}^{m_i} \Lambda_k((\alpha_\ell,\beta_\ell)_{\ell=1}^{m_i},P_{-i})$.

The connection demand of every agent i could consist of m_i pairs of locations $(a_i^k,b_i^k)_{k=1}^{m_i}$ rather than a single pair (a_i,b_i). The characterizations in Theorem 1 and Corollary 1 of simple and super simple allocation rules as well as the characterization in Theorem 2 of allocation rules that truthfully implements MCCNs remain valid. Indeed our proofs of Theorems 1 and 2 and Corollary 1 remain valid without any changes.

6 Final remarks

For the CN model with undemanded locations: we have characterized allocation rules satisfying IUI and NI as well as SI, IUI and NI; we have shown that if an allocation rule implement MCCNs truthfully, then it satisfies IUI and NI; and, we have characterized allocation rules satisfying SI, IUI and NI and implementing MCCNs truthfully. To our surprise it turned out that an allocation rule satisfies SI, IUI and NI if and only if relative cost shares depend on connection demands and nothing else. This was surprising for us because for the MCST model both the folk rule and the family of obligations rules satisfy SI, IUI and NI. Consequently voluntarily participation and Individual Rationality are at odds in the CN model in contrast with the MCST model.

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