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Abstract

This article investigates progressive development of Aumann-Shapley cost allocation in a multilevel organizational or production structure. In particular, we study a linear parametric programming setup utilizing the Dantzig-Wolfe decomposition procedure.

Typically cost allocation takes place after all activities have been performed, for example by finishing all outputs. Here the allocation is made progressively with suggestions for activities. In other words cost allocation is performed in parallel for example with a production planning process. This development does not require detailed information about some technical constraints in order to make the cost allocation.

Keywords: Cost Allocation. Aumann-Shapley Prices. Decomposition. Parametric Programming.

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1 Introduction

It is well recognized that decomposition of linear programs goes beyond computational efficiency and also provides a rationale for decentralized decision making in organizations, see e.g., Dantzig and Wolfe (1960), Baumol and Fabian (1964) and the subsequent planning literature related to multilevel systems such as e.g., Dirickx and Jennergren (1979), Burton and Obel (1984).

In the present paper we aim to link decomposition, and thereby decentralized decisions with limited information, to cost allocation issues within the firm. In particular we consider cost allocation in case of joint production using Aumann-Shapley prices, see e.g., Hougaard (2009). But unlike previous full information models as found e.g., in Mirman, Tauman and Zang (1985) and Hougaard and Tind (2009), we presently assume an underlying decentralized or multilevel organization with limited information.

The main goal here is to emphasize that full information about technical coefficients of the programming problem is *not* required in order to perform a cost allocation using Aumann-Shapley pricing.

As such the paper tries to bridge two important strands of literature: The OR/MS literature on decomposition of linear programs and related decentralized multilevel planning and the economic/game theoretic literature on cost allocation in joint production using Aumann-Shapley pricing.

Examples of joint production in multilevel organizations are abundant. An instance could be that some intermediate parts are produced by subunits which are subsequently transferred to an upper level or central unit to be used in further production. Another example could be an organization with a natural separation between an input and output unit, or production and sales unit.

We consider a general production planning setup with given requirements of outputs and limited capacities in inputs. For a wide range of such models and practical applications, see e.g., Danø (1974). We analyze a joint production process by means of some activities and the corresponding model is a standard activity analysis model in the framework of linear programming. The constraints may naturally be decomposed into two groups, one dealing with the output requirements and the other with the input capacities. In some cases detailed and full information about input consumption and output production is directly available simultaneously. In other cases information about both groups is separated. We consider the situation where information about the output constraints is given in advance, but information about the input constraints is unknown at the outset.

We show that the classical decomposition techniques known from linear programming provides, in a sequential manner, a sufficient part of the information needed in order to make a complete cost allocation using Aumann-Shapley pricing.

The paper is organized as follows: Section 2 recalls the definition of the Aumann-Shapley cost allocation rule in the context of a general cost function. Section 3 presents the full information linear programming model and demonstrates by a numerical example how to apply the AS-allocation rule in that case. Section 4, then moves to the limited information scenario and show how Dantzig-Wolfe decomposition can be used to perform cost allocation progressively as a successive procedure between subunits and a central planning unit. Again the procedure is demonstrated using a numerical example. Section 5 closes with final remarks.

2 Aumann-Shapley Cost Allocation

We start out by recalling the definition of Aumann-Shapley cost allocation for a general cost function.

Consider a joint production process resulting in m different outputs and let $b \in \mathbf{R}^m_+$ be a (non-negative) output vector where b_i is the level of output i. Let the cost of producing any (output) vector b be given by a non-decreasing continuously differentiable cost function $C : \mathbf{R}^m_+ \to \mathbf{R}$ where $C(0) = 0$ (i.e. no fixed costs). Denote by $\partial_i C(b)$ the first order derivative of C at b with respect to the ith argument.

For a given problem (b, C) define the Aumann-Shapley cost allocation (Aumann and Shapley 1974) by cost shares,

$$z_i^{AS}(b,C) = \int_0^{b_i} \partial_i C(\frac{t}{b_i} b)dt = b_i \int_0^1 \partial_i C(tb)dt \quad \text{for all } i = 1, \dots, m. \qquad (1)$$

Using Aumann-Shapley cost allocation results in budget-balance since $\sum_{i=1}^m z_i^{AS}(b,C) = C(b)$.

In particular, $p_i^{AS} = \int_0^1 \partial_i C(tb)dt$ can be seen as the unit cost of output i, also known as the Aumann-Shapley price (A-S price) of output i.

The Aumann-Shapley allocation (A-S rule) can be seen as one (of several) possible extensions of average cost sharing to the multiple product case, see e.g. Hougaard (2009). Axiomatic characterizations are provided (independently) in Billera and Heath (1982) and Mirman and Tauman (1982). Early examples of application can be found in Billera, Heath and Raanan (1978) and Samet, Tauman and Zang (1983). More recent applications include Castano-Pardo and Garcia-Diaz (1995), Haviv (2001) and Babusiaux and Pierru (2007).

Example 1: Consider the simple case where the cost function is homogeneous of degree k, i.e., $C(tb) = t^k C(b)$ for $t \in [0, 1]$. Here it is clear that for all $i \in N$, the A-S prices become

$$p_i^{AS} = \partial_i C(q) \int_0^1 t^{k-1} dt = p_i^{MC} \frac{1}{k},$$

and hence,

$$z_i^{AS}(b, C) = b_i p_i^{MC} \frac{1}{k},$$

where p_i^{MC} is the marginal cost price of i. △

In the following we shall formulate the cost structure as a linear programming problem along the lines of Samet, Tauman and Zang (1984), Mirman, Tauman and Zang (1985) and Hougaard and Tind (2009).

3 Cost Allocation with Full Information

For later reference we shall first consider the standard full information case. We formulate a linear programming model that minimizes overall costs and at the end our procedure suggests a natural allocation of these costs onto the individual outputs of production by way of A-S cost allocation.

The problem may be interpreted as a cost minimization problem for a production made by some activities, where $x \in \mathbf{R}^n$ denotes the activity levels. $b \in \mathbf{R}^m$ is a vector of required outputs. $A \in \mathbf{R}^{m \times n}$ is a matrix of unit production coefficients of the activities where $Ax \geq b$ ensures sufficient production by means of the activities. On the other hand let $B \in \mathbf{R}^{p \times n}$ be a matrix of unit consumption coefficients for the activities using some capacities. Let $d \in \mathbf{R}^p$ denote the capacity limits. Introduce the constraints $Bx \leq d$ to be interpreted as capacity constraints. Finally let $c \in \mathbf{R}^n$ denote the unit cost coefficients of the activities.

This leads to the consideration of the following linear programming problem

$$\begin{aligned} z_0 = \min \quad & cx \\ \text{s.t.} \quad & Ax \geq b \\ & Bx \leq d \\ & x \geq 0. \end{aligned} \tag{2}$$

The objective calculates the minimal total costs z_0. The question is how to allocate these costs onto each of the output elements.

Let $u \in \mathbf{R}^m$ denote the optimal dual variables corresponding to the output vector b and let $v \in \mathbf{R}^p$ denote the optimal dual variables corresponding to the capacities d. By the duality theorem in linear programming $z_0 = ub + vd$. Let u_i denote the elements of u and let b_i denote the elements of b for $i = 1, \ldots, m$. The term $u_i b_i$ allocates thus only partly some costs to each output element. The remaining term vd can be interpreted as the costs related to the capacities. Hence the fundamental question becomes how to allocate vd on to the output elements in a meaningful way.

Here the A-S rule suggests a useful approach utilizing the so-called parametric programming approach known from linear programming. For the general parametric programming approach see for example Vanderbei (2008). Consider the problem (2) in the following parameterized form.

$$\begin{aligned} z(t) = \min \quad & cx \\ \text{s.t.} \quad & Ax \geq tb \\ & Bx \leq d \\ & x \geq 0 \end{aligned} \tag{3}$$

where $t \in [0, 1]$ is a scalar.

The objective function is piecewise linear and concave in t. We assume that the problem is feasible for all t. In an economic context it is also reasonable to let all data be non-negative. This is not an important assumption for the theoretical development, but it implies the natural property that the objective function is non-decreasing in the output, and that $z(0) = 0$ with no output. Let $t_1, \ldots, t_r$ denote the break points in the linearity of $z(t)$ in the interval [0,1]. Within each linear interval between two consecutive break points the optimal dual variables of the production constraints $Ax \geq tb$ are constant and equal to the gradient of the objective function in (3) with respect to tb. Let u_i^k denote the dual variables in the interval $[t_k, t_{k+1}]$ for $i = 1, \ldots, m$ and $k = 1, \ldots, r-1$. For each interval the A-S rule calculates the differences $(t_k - t_{k-1})u_i^k b_i$. For each output element i we add all the differences and obtain $z_i = \sum_{k=2}^{r}(t_k - t_{k-1})u_i^k b_i$. Finally observe by construction that $\sum_{i=1}^{m} z_i = z_0$. In this way the A-S rule gives by means of the elements z_i an allocation of all costs including the capacity costs on to each output element i.

Example 2: Consider the following problem,

$$\begin{array}{lrcrcrcl} \min & 6x_1 & + & 7x_2 & + & 2x_3 & & \\ & 9x_1 & + & 12x_2 & + & 6x_3 & \geq & 16 \\ & 7x_1 & + & 14x_2 & + & 3x_3 & \geq & 14 \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3. \end{array} \tag{4}$$

The output vector is here $b = (16, 14)$. The optimal objective value of the above program is $9\frac{8}{15} = 9.53$. We shall allocate this cost on to the elements of the output vector according to the A-S rule. We therefore consider the parameterized program (3) which undertakes the form:

$$\begin{array}{lrcrcrcl} \min & 6x_1 & + & 7x_2 & + & 2x_3 & & \\ & 9x_1 & + & 12x_2 & + & 6x_3 & \geq & 16t \\ & 7x_1 & + & 14x_2 & + & 3x_3 & \geq & 14t \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3. \end{array}$$

The parametric analysis results in three intervals with values as shown by the following table.

k	t	u_1^k	u_2^k
1	$0 \leq t \leq \frac{10}{13}$	$\frac{7}{48}$	$\frac{3}{8}$
2	$\frac{10}{13} \leq t \leq \frac{95}{98}$	$\frac{5}{6}$	$\frac{1}{10}$
3	$\frac{95}{98} \leq t \leq 1$	$\frac{23}{15}$	0

table 1

From the above table we get the Aumann-Shapley allocated costs

$$z_1^{AS} = 16 \times \{\frac{10}{13} \times \frac{7}{48} + (\frac{95}{98} - \frac{10}{13}) \times \frac{5}{6} + (1 - \frac{95}{98}) \times \frac{23}{15}\} = 5.21$$

$$z_2^{AS} = 14 \times \{\frac{10}{13} \times \frac{3}{8} + (\frac{95}{98} - \frac{10}{13}) \times \frac{1}{10} + (1 - \frac{95}{98}) \times 0\} = 4.32$$

.

Observe that the A-S cost shares result in budget-balance, i.e., $z_1^{AS} + z_2^{AS} = 9.53$. This will not be the case if we used marginal cost pricing instead:

Indeed, the marginal costs for $t = 1$ are equal to the optimal dual variables of the original program (4), i.e., $p_i^{MC} = u_i^3$ for $i = 1, 2$. These do not yield a reasonable cost allocation, in particular because $u_2^3 = 0$ and therefore assigns nothing to the second output. Moreover $u_1^3 b_1 + u_2^3 b_2 = \frac{23}{15} \times 16 + 0 \times 14 = 24.53$, which exceeds the total costs and therefore results in a surplus. △

4 Cost Allocation with limited information

The constraints of the main program (2) may naturally be decomposed into two parts, one dealing with the outputs and the other with the capacities. We shall therefore examine the problem from a decomposition point of view and focus on the now classical relationship between the decomposition procedure in linear programming and its interpretation in decentralized planning of a decomposed multilevel organization or production structure.

The main idea is that some data are unknown from the outset. Here we assume that the capacity constraints $Bx \leq d$ are unknown. In an economic context this makes sense for a multilevel organizational or production structure where detailed information about capacity or production possibilities is unknown for the central planning unit and only directly available for one or more subunits. During an iterative process sufficient information from the subunit level is transferred until optimality is proven. We here use the classical Dantzig-Wolfe decomposition method noting that the set $\{x \geq 0 | Bx \leq d\}$ is polyhedral and thus may be described as the convex hull of its extreme points (and possibly by additional extreme rays). The main idea is that only a limited number of extreme points is needed in order to prove optimality. For detailed information about the decomposition procedure in linear programming see e.g., Dantzig and Thapa (2003).

The procedure progresses in iterations generating extreme points when needed. Let p denote the number of extreme points generated at a certain stage, and let x^l denote an extreme point for $l = 1, \ldots, p$. Introduce the scalar t with a preliminary fixed value. According to the Dantzig-Wolfe decomposition procedure we shall consider the so-called master program:

$$\begin{aligned} \min \quad & \sum_{l=1}^{p} cx^l \lambda_l \\ \text{s.t.} \quad & \sum_{l=1}^{p} Ax^l \lambda_1 \geq tb \\ & \sum_{l=1}^{p} \lambda_l \leq 1 \\ & \lambda_l \geq 0 \text{ for } l = 1, \ldots, p, \end{aligned} \tag{5}$$

where $\lambda_l \in \Re$ is a variable for $l = 1, \ldots, p$.

Solve the master program (5) and let $u \in \Re^m$ be the optimal dual variables corresponding to the output vector b. Moreover, let v denote the optimal dual variable corresponding to the convexity constraint $\sum_{l=1}^{p} \lambda_l \leq 1$. (Usually $\sum_{l=1}^{p} \lambda_l = 1$. However this equation is here changed into an inequality noting that 0 is an extreme point by assumption, corresponding to the instance of no production.) Consider next the so-called subproblem:

$$\begin{aligned} z_{sub} = \min \quad & (c - uA)x \\ \text{s.t.} \quad & Bx \leq d \\ & x \geq 0. \end{aligned} \tag{6}$$

Let x^l denote the optimal solution of (6). If $z_{sub} - v < 0$ then x^l shall be included in the set of extreme points in the master program (5) and the number p of extreme points is increased by 1. If $z_{sub} - v = 0$ the procedure stops and an optimal solution $x = \sum_{l=1}^{p} \lambda_l x_l$ of (3) has been found by the preceeding master program.

There is no need to make a complete analysis of (5) for all $t \in [0, 1]$. As long as the dual variables from (5) generate negative values of $z_{sub} - v$ for some t the master problem (5) is extended with a new column and the procedure continues. Otherwise a parametric analysis of t must be performed until $z_{sub} - v = 0$ for all $t \in [0, 1]$.

A each stage of the master program we may perform cost allocation using the A-S rule in the same way as in the case of full information. In this way the decomposition leads to a progressive development of the A-S cost shares ending up with the same result as in the full information case.

Example 2 continued: We shall consider the same example as studied in Example 2 above. We initiate the procedure here by selecting some arbitrary dual variables to be inserted in the first subproblem. We let $(u_1, u_2) = (1, 1)$ and the subproblem becomes

$$\begin{array}{rrcrcrcl} z_{sub} = \min & -10x_1 & - & 19x_2 & - & 7x_3 & & \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3. \end{array}$$

with optimal solution $(x_1, x_2, x_3) = (\frac{5}{14}, \frac{15}{14}, 0)$ and $z_{sub} - v = -\frac{335}{14} - 0 < 0$. Hence this optimal solution is inserted as an extreme point creating the first column in the master problem, which becomes

$$\begin{array}{rrcl} \min & \frac{135}{14}\lambda_1 & & \\ & \frac{225}{14}\lambda_1 & \geq & 16t \\ & \frac{245}{14}\lambda_1 & \geq & 14t \\ & \lambda_1 & \leq & 1 \\ & \lambda_1 & \geq & 0. \end{array}$$

with an optimal dual solution $(u_1, u_2, v) = (\frac{3}{5}, 0, 0)$ (for any $t \in [0, 1]$). The next subproblem becomes

$$\begin{array}{rrcrcrcl} z_{sub} = \min & \frac{3}{5}x_1 & - & \frac{1}{5}x_2 & - & \frac{8}{5}x_3 & & \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3. \end{array}$$

with optimal solution $(x_1, x_2, x_3) = (0, 0, \frac{5}{3})$ and $z_{sub} - v = -\frac{8}{3} - 0 < 0$. Hence this optimal solution is inserted as an extreme point creating the second column in the master problem, which becomes

$$\begin{array}{rrcrcl} \min & \frac{135}{14}\lambda_1 & + & \frac{10}{3}\lambda_2 & & \\ & \frac{225}{14}\lambda_1 & + & 10\lambda_2 & \geq & 16t \\ & \frac{245}{14}\lambda_1 & + & 5\lambda_2 & \geq & 14t \\ & \lambda_1 & + & \lambda_2 & \leq & 1 \\ & & & \lambda_j & \geq & 0 \text{ for } j = 1, 2. \end{array}$$

With $t = 0.1$ we get an optimal dual solution $(u_1, u_2, v) = (\frac{17}{159}, \frac{24}{53}, 0)$. The following subproblem is

$$\begin{array}{rrcrcrcl} z_{sub} = \min & \frac{99}{53}x_1 & - & \frac{33}{53}x_2 & + & 0x_3 & & \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3. \end{array}$$

$(x_1, x_2, x_3) = (0, \frac{5}{4}, 0)$ is an optimal solution and $z_{sub} - v = -\frac{165}{212} - 0 < 0$. Hence this optimal solution is inserted as an extreme point creating the third column in the next master problem

$$\begin{array}{rrcrcrcl} \min & \frac{135}{14}\lambda_1 & + & \frac{10}{3}\lambda_2 & + & \frac{35}{4}\lambda_3 & & \\ & \frac{225}{14}\lambda_1 & + & 10\lambda_2 & + & 15\lambda_3 & \geq & 16t \\ & \frac{245}{14}\lambda_1 & + & 5\lambda_2 & + & \frac{35}{2}\lambda_3 & \geq & 14t \\ & \lambda_1 & + & \lambda_2 & + & \lambda_3 & \leq & 1 \\ & & & & & \lambda_j & \geq & 0 \text{ for } j = 1, \ldots, 3. \end{array}$$

With $t = 0.1$ we get the optimal values $(u_1, u_2, v) = (\frac{7}{48}, \frac{3}{8}, 0)$ and the subproblem

$$\begin{array}{rrcrcrcl} z_{sub} = \min & \frac{33}{16}x_1 & + & 0x_2 & + & 0x_3 & & \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3. \end{array}$$

$(x_1, x_2, x_3) = (0, 0, 0)$ is an optimal solution and $z_{sub} - v = 0 - 0 = 0$ and no new column is going to be created for the master problem.

Hence it is time to perform a parametric analysis of the last master problem. Calculation shows that the optimal duals are unchanged for $0 \leq t \leq \frac{10}{13}$.

For $\frac{10}{13} \leq t \leq \frac{95}{98}$ we get optimal values $(u_1, u_2, v) = (\frac{5}{6}, \frac{1}{10}, -\frac{11}{2})$ and with these values inserted we get the subproblem

$$\begin{array}{rrcrcrcl} z_{sub} = \min & -\frac{11}{5}x_1 & - & \frac{22}{5}x_2 & - & \frac{33}{10}x_3 & & \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3. \end{array}$$

$z_{sub} - v = -\frac{11}{2} + \frac{11}{2} = 0$ and no new column is going to be inserted the the masterproblem.

For $\frac{95}{98} \leq t \leq 1$ we get that $(u_1, u_2, v) = (\frac{23}{15}, 0, -15)$ is an optimal solution in the previous masterproblem. With those values we get the subproblem

$$\begin{array}{rrcrcrcl} z_{sub} = \min & -\frac{39}{5}x_1 & - & \frac{57}{5}x_2 & - & \frac{36}{5}x_3 & & \\ & 12x_1 & + & 10x_2 & + & 3x_3 & \leq & 15 \\ & 4x_1 & + & 8x_2 & + & 6x_3 & \leq & 10 \\ & & & & & x_i & \geq & 0 \text{ for } i = 1, \ldots, 3 \end{array}$$

and the optimal value $z_{sub} - v = -15 + 15 = 0$ and no new column is going to be inserted. The analysis is completed and from the results above we recover the contents of table 1 in Example 2, and thereby also the same A-S cost shares as in the full information case. △

As demonstrated above we have shown that the transfer and allocation of costs do not require a complete statement and formulation of the constraints receiving no cost allocation at the end. The procedure, based on classical decomposition, sequentially brings in sufficient information about this set of constraints, and finally all costs are allocated only to those constraints, that were known in advance.

5 Final remarks

The linear programming model considered is rather general. The basic idea is to move (shadow-)prices in the form of optimal dual variables from some constraints on to other constraints in a meaningful way applying some well founded allocation rule.

In this paper we have chosen to focus on the Aumann-Shapley rule. Other rules may be applied as well using exactly the same techniques. For example we could also have applied the more recently developed Friedman-Moulin ruled (introduced by Friedman and Moulin 1999). As argued in the literature the Friedman-Moulin rule has the advantage of being ”demand monotonic” in the sense that if demand for one output is increased (other demands being equal) the cost share of that output cannot decrease. The A-S rule is not demand monotonic. Yet, the Friedman-Moulin rule requires a meaningful ordering of the outputs which is difficult to interpret given these typically will be measured in different units of measurement.

Finally, we note that the idea of decomposition goes beyond linear programming. A rather extensive literature exists on convex and separable programming, see e.g., Moeseke and Ghellinck (1969), Spingarn (1985). This suggests possibilities for considering cost allocation in optimization models that are no longer linear.

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