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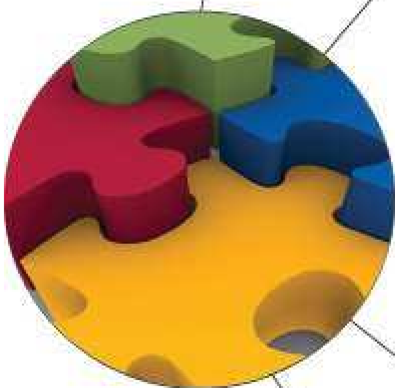

Incremental Cost Sharing in Chains and Fixed Trees

Jens Leth Hougaard* **Mich Tvede†**

Abstract

In this paper we consider a family of cost allocation rules for which agents pay a share of their incremental cost as well as of any 'debt' from prior agents. This family encompasses the Bird rule and the free riding rule (where terminal agents pay everything) as the two extreme cases. By axiomatic characterization it is demonstrated that a distinguishing feature of this family is that payments are independent of both the costs and the number of downstream agents.

Keywords: Cost allocation, Axiomatic characterization, Bird allocation, Incremental cost sharing.

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1 Introduction

The topic of cost allocation in rooted networks (in particular trees) has a long line of history as surveyed e.g., in [14] and [15]. The problem is typically approached by considering either how network structure should influence allocation directly (e.g., [3]) or by reinterpreting the problem as a cooperative game (e.g., [1]).

In the present paper we take the former approach and consider chains and fixed trees. In particular, we investigate a broad family of allocation rules where agents' payments are based on shares of their incremental costs as well as any 'debt' from prior agents - to be dubbed sequential λ-contribution rules. This family encompasses the so-called Bird rule (each agent pays the cost of the adjacent edge in direction of the root) and the free rider rule (terminal agents share the total cost) as extremes.

Rules based on incremental costs are often considered unfair since the stand-alone cost of an agent may be large even though the incremental cost is small (e.g., [6]). Moreover, such rules perform badly in terms of implementation (e.g., [7]). Yet, they result in core allocations so no coalition of agents can actually gain by 'standing alone' (e.g., [6]). In situations where it seems less relevant that cost shares should be increasing in stand-alone costs they constitute simple and attractive alternatives to traditional solution concepts from cooperative game theory. For example, sharing the costs of washing staircases, elevator installation or scaffolding in apartment buildings, or sharing irrigation costs among farmers etc.

We provide an axiomatic characterization of sequential λ-contribution rules and further single out the two extreme rules in the family: the Bird rule and the 'free rider' rule. The essential feature of the sequential λ-contribution rules is that agents' payments are independent of the presence of 'downstream' agents both with respect to their costs and their number. This is in contrast to the well-known serial rule (the Shapley value) where agents cost shares depend on the number of 'downstream' agents.

2 The Chain Model

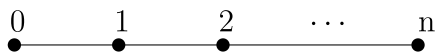
We study cost allocation in a variable population model. The set of agents is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of all subsets of $\mathbb{N}$, with generic element N. Let n denote the cardinality of N.

For a given set of agents $N = \{1, \ldots, n\}$, each agent $i \in N$ is characterized by a (stand-alone) cost C_i. Assume for simplicity that

$$0 < C_1 < \ldots < C_n.$$

Denote by $C = (C_1, \ldots, C_n)$ the profile of stand-alone costs.

The cost structure can be represented by a chain: The cost of the first edge (from the root 0 to agent 1) is C_1, the cost of the second edge (from agent 1 to agent 2) is equal to the incremental cost $C_2 - C_1$ and so forth.



Denote by the pair (N, C) a *cost allocation problem* and let $\mathcal{C}$ be the set of cost allocation problems.

A *cost allocation rule* is a function $\phi : \mathcal{C} \to \mathbb{R}^N$ satisfying: 1. *budget balance*, $\sum_{i \in N} \phi_i(N, C) = C_n$; and, 2. *boundedness*, $0 \leq \phi_i(N, C) \leq C_i$ for all $i \in N$.

This allocation problem captures many practical cases [6]. For instance, the so-called 'airport game' where solutions like the Nucleolus and the Shapley value become operational means to determine airport landing fees, [8], [9], [10], [11].

3 Cost Allocation in Chains

Since there is no externality between agents, an appealing property of allocation rules is that the cost share of agent i is independent of the stand-alone cost of any (downstream) agent j with $C_j > C_i$.

This 'independence' property is satisfied by many well-known rules. For instance, the rule where each agent pays for the increment that she is responsible for, i.e., where cost shares are given by,

$$x_i^B = C_i - C_{i-1}, \tag{1}$$

for all $i \in N$. The allocation defined by (1) is often named the *Bird allocation* but it appears under many different names in the literature, e.g. the *sequential full contributions rule* ([15]).

In this paper we will investigate a broad family of rules based on incremental costs. Consider allocating in the following way: the first $n-1$ agents pay a share $\lambda \in [0,1]$ of their incremental cost as well as any remaining 'debt' from prior agents, and the last agent n pays the residual. Hence, cost shares are determined as $x_1^\lambda = \lambda C_1$, $x_2^\lambda = \lambda(C_2 - C_1 + (1-\lambda)C_1) = \lambda(C_2 - x_1^\lambda)$ and so forth. That is,

$$x_i^\lambda = \lambda(C_i - \sum_{j=1}^{i-1} x_j^\lambda), \tag{2}$$

for all $i = 1, \dots, n-1$ and $x_n^\lambda = C_n - \sum_{j=1}^{n-1} x_j^\lambda$.

Denote the corresponding family of allocation rules by ϕ^λ. Clearly, ϕ^1 is the sequential full contributions rule yielding the allocation defined in (1). The opposite extreme, ϕ^0, represents the case where the first $n-1$ agents free ride and the last agent pays the total cost.

A popular allocation rule in the literature is the rule where all downstream agents share equally each incremental cost, i.e., cost shares are given by,

$$x_i^S = \sum_{k=1}^{i} \frac{C_k - C_{k-1}}{n+1-k}. \tag{3}$$

This rule appears under many different names in the literature, e.g. the *sequential equal contributions rule* ([15]). This rule coincides with the Shapley value of a TU-game (N, c) defined by $c(S) = \max_{i \in S}\{C_i\}$, for all $S \subseteq N$. Axiomatic characterizations can be found in [4], [12], [2] and [13].

We observe that payments of the sequential equal contribution rule (3) depend on the *number* of downstream agents in contrast to payments of the sequential λ-contribution rules (2). This aspect will be further explored in the section below.

4 Axiomatics

We shall now provide an axiomatic characterization of the family of allocation rules ϕ^λ.

The agent with the maximal stand-alone cost plays a special role in the cost allocation problem because he determines the total cost of the group. Yet, if an agent is not the largest agent it can be argued that such an agent's cost share should dependent only on the costs of the (unique connected) path from the agent to the root. Formally,

Independence of Downstream Agents (IDA): For two cost allocation problems (N, C) and (N', C') with $\{1, \ldots, m+1\} \subset N \cap N'$ and $C'_i = C_i$ for all $i \leq m$

$$\phi_i(N', C') \;=\; \phi_i(N, C)$$

for all $i \leq m$.

In other words, if two cost allocation problems coincide for a lower coalition of agents with no agent being the last agent, then these agents should pay the same. IDA is clearly satisfied by the sequential full contributions rule (as well as the rest of the λ-contribution rules, ϕ^λ), but it is *not* satisfied by the sequential equal contributions rule because the cost shares depend on the total number of agents in the chain.

For the next axiom we need to define a reduced cost profile. Consider the cost profile C and let $\phi(N, C)$ be the associated cost allocation. Then excluding agent 1, results in a new cost profile C^ϕ_{-1} for the reduced problem where agent 1's cost share, $\phi_1(N, C)$, is subtracted from the stand-alone cost of all the remaining agents, i.e.,

$$C^\phi_{-1} = (C_2 - \phi_1(N, C)), \ldots, C_n - \phi_1(N, C))).$$

We now require that the allocation among agents $\{2, \ldots, n\}$ is unchanged by exclusion of agent 1. Formally,

First-Agent Consistency (FAC): For all cost allocation problems (N, C)

$$\phi_i(N \setminus \{1\}, C^{\phi}_{-1}) \;=\; \phi_i(N, C)$$

for all $i \geq 2$.

This axiom is analyzed in [13]. It is shown that several rules satisfies FAC, e.g., both the sequential equal- and the sequential full contributions rule, and by Theorem 2.4 in [13], all rules satisfying FAC result in core allocations.

Finally, we consider the property of scale invariance stating that if stand-alone costs are scaled by a factor α, so is the solution.

Scale Invariance (SI): For all cost allocation problems (N, C)

$$\phi(N, \alpha C) \;= \alpha\phi(N, C)$$

for all $\alpha > 0$.

Scale invariance is a relatively mild condition which is satisfied by most well known rules, e.g. both the sequential equal- and the sequential full contributions rule.

Theorem 1: *A cost allocation rule ϕ satisfies IDA, FAC and SI if and only if $\phi(N, C) = \phi^{\lambda}(N, C)$.*

Proof: It is easy to see that ϕ^{λ} satisfies IDA, FC and SI.

Consider the reverse statement. First, let $N = \{1\}$ and $C = C_1$. By budget balance, $\phi_1(N, C) = C_1 = \phi^{\lambda}_1(N, C)$. Next, add a second agent with $C_2 > C_1$. Let $N' = \{1, 2\}$ and $C' = (C_1, C_2)$. Then $\phi_1(N', C') \in [0, C_1]$ so $\phi_1(N', C') = \lambda C_1 = \phi^{\lambda}_1(N', C')$ for some $\lambda \in [0, 1]$. By IDA, λ is independent of C_2 and by SI, λ is independent of C_1. By budget balance, $\phi_2(N', C') = C_2 - \phi_1(N', C') = \phi^{\lambda}_2(N', C')$. Next, add a third agent with $C_3 > C_2$. Let $N'' = \{1, 2, 3\}$ and $C'' = (C_1, C_2, C_3)$. By IDA, $\phi_1(N'', C'') = \phi_1(N', C') = \lambda C_1 = \phi^{\lambda}_1(N'', C'')$. By FAC, $\phi_2(N'', C'') = \phi_2(N'' \setminus \{1\}, C''^{\phi}_{-1})$ and by IDA and SI, $\phi_2(N'' \setminus \{1\}, C''^{\phi}_{-1}) = \lambda(C_2 - \phi_1(N'', C'')) = \phi^{\lambda}_2(N'', C'')$. By budget balance $\phi_3(N'', C'') = C_3 - \phi_2(N'', C'') - \phi_1(N'', C'') = \phi^{\lambda}_3(N'', C'')$.

Repeated use of the axioms gives the desired conclusion that $\phi(N,C) = \phi^\lambda(N,C)$.

Q.E.D.

Remark 1: Concerning independence of the axioms: 1. the sequential equal contributions rule satisfies FAC and SI, but not IDA; 2. ϕ^λ, where $\lambda = C_1/(1+C_1)$, satisfies IDA and FAC, but not SI; and, 3. the rule for which $\phi_i(N,C) = \lambda(C_i - C_{i-1})$ for $i = 1,\ldots,n-1$ (where $C_0 = 0$ and $0 < \lambda < 1$) and $\phi_n(N,C) = C_n - \sum_{j=1}^{n-1}\phi_j(N,C)$, satisfies IDA and SI, but not FAC.

The sequential full contributions rule ϕ^1 can be singled out by replacing FAC and SI with a property stating that cost shares should be increasing in edge costs. Formally,

Edge Cost Ranking (ECR): For every cost allocation problem (N,C)

$$\phi_j(N,C') \;\geq\; \phi_k(N,C)$$

for all j and k with $C_j - C_{j-1} \geq C_k - C_{k-1}$.

Corollary 1: *A cost allocation ϕ satisfies IDA and ECR if and only if it is the sequential full contributions rule $\phi = \phi^1$.*

Proof: It is easy to see that ϕ^1 satisfies IDA and ECR.

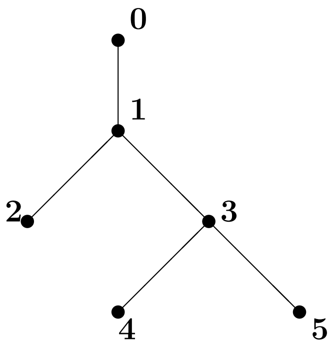
Consider the reverse statement. First, let $N = \{1\}$ and $C = C_1$. By budget balance, $\phi_1(N,C) = C_1 = \phi_1^1(N,C)$. Next, add a second agent with $C_2 > C_1$. Let $N' = \{1,2\}$ and $C' = (C_1, C_2)$. Then $\phi_1(N',C') \in [0, C_1]$ so $\phi_1(N',C') = \lambda C_1 = \phi_1^\lambda(N',C')$ for some $\lambda \in [0,1]$. By IDA, λ is independent of C_2. Now, assume that $\lambda < 1$. Then by budget balance we may choose C_2 such that ECR is violated. Thus $\lambda = 1$ as desired. By repeated use of IDA we are done.

Q.E.D.

Remark 2: The opposite extreme of ϕ^1 in the λ-family, ϕ^0, may be singled out by replacing FAC with a ranking property stating that payments are increasing in stand-alone costs.

5 The Fixed Tree Model

The family of allocation rules ϕ^λ has a natural extension to cost allocation in fixed trees. A fixed tree is a weighted rooted tree where we assume that each node is occupied by one agent and each edge is weighted by its cost. Only one edge leaves the root as exemplified by the illustration below.



For a given set of agents (nodes) N, denote by c_i the cost of the edge leaving the node i on agent i's unique connected path to the root. The total cost of the fixed tree therefore is $C = \sum_{i \in N} c_i$. Let $c = (c_1, \ldots, c_n)$ be the profile of edge costs. The pair (N, c) constitutes a (fixed tree) cost allocation problem. Let $\mathcal{F}$ be the set of (fixed tree) cost allocation problems.

As before, a *cost allocation rule* on the set of cost allocation problems is a function $\phi : \mathcal{F} \to \mathbb{R}^N$ satisfying: 1. *budget balance*, $\sum_{i \in N} \phi_i(N, c) = C$; and 2. *boundedness*, $0 \leq \phi_i(N, c) \leq \sum_{i \in N_i} c_i$ for all $i \in N$, where $N_i \subseteq N$ is the set of nodes on agent i's unique connected path to the root (including node i itself).

Clearly, the chain is a special case of a fixed tree with edge cost profile $c = (C_1, C_2 - C_1, \ldots, C_n - C_{n-1})$.

Let $\delta(i)$ be the degree of node i (i.e., the number of nodes adjacent to node i). Consider a node $i \in N$, if $\delta(i) = 1$, node i is called a *terminal node*.

The family of allocation rules ϕ^λ easily extends to fixed trees: Agents that are not at terminal nodes pay a share $\lambda \in [0, 1]$ of their edge cost c_i as well as an equal share (between successors of the node prior to i) of any remaining 'debt' and agents at terminal nodes pay their respective residuals. Abusing notation we say that nodes $\{1, \ldots, i-1\}$ are the nodes prior to node i on i's path to the root.

Since $\delta(0) = 1$ we get,

$x_1^\lambda = \lambda c_1$,

and for other agents, i, not at terminal nodes, we get
$x_i^\lambda = \lambda(c_i + \frac{c_{i-1}-x_{i-1}^\lambda}{\delta(i-1)-1} + \frac{c_{i-2}-x_{i-2}^\lambda}{(\delta(i-2)-1)(\delta(i-1)-1)} + \ldots + \frac{c_1-x_1^\lambda}{\prod_{z=1}^{i-1}\delta(z)-1})$.

For agents, i, at terminal nodes, we get,
$x_i^\lambda = c_i + \frac{c_{i-1}-x_{i-1}^\lambda}{\delta(i-1)-1} + \frac{c_{i-2}-x_{i-2}^\lambda}{(\delta(i-2)-1)(\delta(i-1)-1)} + \ldots + \frac{c_1-x_1^\lambda}{\prod_{z=1}^{i-1}\delta(z)-1}$.

Again, $\lambda = 1$ gives us the sequential full contributions rule (the *Bird allocation* $x_i^1 = c_i$ for all i) and $\lambda = 0$ gives the rule where agents at terminal nodes pay the relevant residual.

In terms of axiomatic characterization it is easy to extend the axioms of the chain model: An agent j is a downstream agent for an agent i if and only if i is prior to j on j's path to the root. IDA now states that the cost share of agent i only depends on whether the sub-tree of downstream agents is empty or not, but not on the structure or the edge costs of the sub-tree.

In order to extend First-Agent Consistency (FAC), the reduced edge cost profile needs to be defined. Clearly, excluding agent 1, results in $\delta(1) - 1$ new fixed trees. The reduced edge cost profile is defined as follows: for an immediate successor of agent 1 the edge cost is changed to $c_i - (1/(\delta(1) - 1))\phi_1(N, c)$ and for all other agents the edge cost is unchanged c_i.

We now state the following result.

Corollary 2: *The result of Theorem 1 extends to the fixed tree model with the above restatements of the axioms.*

Sketch of proof: For agent 1 it does not matter whether the successors form a chain or some other tree according to IDA. Therefore the proof of Theorem 1 applies. Consequently, agent 1 pays λC_1 for some $\lambda \in [0, 1]$. Using FAC to remove agent 1, results in $\delta(1) - 1$ new trees. Again the proof of Theorem 1 applies to each of the new trees. Consequently the immediate successors of agent 1 pay λ times their own edge costs as well as the debt from agent 1 because of IDA and SI.

End of sketch

Remark 3: The two ways to single out the extreme rules ϕ^1 and ϕ^0, mentioned in Corollary 1 and Remark 2 respectively, extends in a similar fashion to the fixed tree model.

Remark 4: As shown in [5] the cooperative game related to the fixed tree (where $c(S)$ is defined as the minimal cost needed to join all agents in coalition $S \subset N$ to the root via a connected subgraph of the fixed tree) is concave so the core is relatively large. Obviously, ϕ^λ results in core allocations for all $\lambda \in [0,1]$.

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