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Abstract

In this paper we consider the Minimum Cost Spanning Tree model. We assume that a central planner aims at implementing the minimum cost spanning tree not knowing the true link costs. The central planner sets up a game where agents announce the link costs, a tree is chosen and costs are allocated according to the rules of the game. We characterize ways of allocating costs such that true announcements constitute Nash equilibria. In particular, we find that the Shapley rule with respect to the irreducible cost matrix is consistent with truthful announcements while a series of other well-known rules (such as the Bird-rule, Serial Equal Split, the Proportional rule etc.) are not.

Keywords: Minimum cost spanning tree, Strategyproof Implementation, Nash equilibrium, Shapley value.

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1 Introduction

Recently, economists have shown a growing interest in networks and the by now substantial literature is rich on various issues and models, see e.g. Goyal (2007) and Jackson (2008). In the present paper we consider the relation between cost allocation and efficient network structure. In particular, we analyze the classical minimal minimum cost spanning tree model where a group of agents is connected to a source (supplier) in the least costly way and face the problem of sharing the cost of the efficient network, see e.g. Sharkey (1995) – practical examples include district heating, computer network using a common server, cable tv, chain stores using a common warehouse etc.

While the literature typically contains axiomatic analysis and comparisons of different cost sharing methods there has been less emphasis on strategic issues concerning practical implementation. Clearly, it is easy to imagine a series of practical applications where strategic reactions can be expected among the agents involved. Indeed, agents may have private information about the cost structure, which they can use strategically to lower their own cost at the expense of a loss in social efficiency.

It is well-known that in general we cannot find incentive compatible and efficient cost allocation mechanisms (Green and Laffont 1977), but under special circumstances such mechanisms can in fact be constructed, see e.g. Jackson and Moulin (1992), Schmeidler and Tauman (1994), Young (1998), Moulin and Shenker (2001).

In the specific context of the minimum cost spanning tree model, implementation has been analyzed in a few recent papers; Bergantinos and Lorenzo (2004, 2005) and Bergantinos and Vidal-Puga (2008). All three papers consider existence and properties of Nash equilibria and subgame perfect Nash equilibria of non-cooperative sequential bargaining procedures. The first two papers study a real life allocation problem where agents sequentially join an existing network along the lines of the Prim algorithm. The latter paper shows that another Prim-like procedure, where agents announce their willingness to pay for other agents to connect to the source, leads to a unique subgame perfect Nash equilibrium. In this equilibrium, costs are allocated corresponding to the use of a rule coinciding with the Shapley value on the related irreducible cost matrix (dubbed the Folk-solution in Bogomolnaia and Moulin 2008 and further analyzed in Bergantinos and Videl-Puga 2007, 2009).

In the present paper we analyze yet another implementation scenario of the minimum cost spanning tree model: Agents each announce the link costs to a central planner who then based on these announcements estimates the cost matrix, selects a cost minimizing spanning tree given the estimated link costs and then allocates the actual (true) costs of the selected spanning tree according to a prespecified allocation rule. Thus, agents' announcements about the link costs only influence the planners selection of the efficient spanning tree. Obviously, agents may lie about the link costs but once the "efficient" spanning tree has been selected the planner gets to know the actual costs of the links in the selected tree. In turn, these true costs are used to determine the final cost allocation. Motivated by the observation that agents need not have information about the other agents' link costs we further consider a scenario where agents are restricted to announce their own link costs only.

Compared to the Prim-like sequential mechanisms mentioned above our approach is different since it is a simple one-shot game, which is not based on any algorithm for finding the minimum cost spanning tree. Moreover, agents announcements do not directly influence their cost shares since these are determined by the realized costs along the chosen spanning tree (which is selected according to the announced link costs).

As our main result we show that announcing the true link costs constitutes a Nash equilibrium if and only if the associated cost allocation rule is monotonic in the sense that all cost shares are weakly increasing in the (irreducible) link cost matrix. Consequently, monotonic allocation rules such as the Shapley rule (on the irreducible cost matrix) and the Equal Split rule will both result in strategyproof Nash equilibria where the planner can implement the true minimum cost spanning tree[1]. Meanwhile, well-known rules such as the Bird rule and the Proportional rule fail to satisfy monotonicity

[1]It turns out that the Equal Split rule also plays an important role implementing the efficient graph in the more general framework of connection networks, Kumar and Juarez (2009).

as demonstrated by simple examples. We further find a class of rules (satisfying Responsiveness) for which the main result carries over to the restricted implementation game.

The paper is organized as follows; Section 2 reviews the minimum cost spanning tree model as well as various well-known allocation rules, defines the irreducible cost matrix and prove an important Lemma. Section 3 defines our implementation game and Section 4 is devoted to results including our main characterization result. Section 5 considers a restricted model where the agents only announce their own link costs and demonstrates the extend to which the main characterization carries over to this scenario. Section 6 closes with final remarks.

2 The MCST Model

We recall the well known minimum cost spanning tree model (see e.g. Sharkey 1995). Let $N = \{1, \ldots, n\}$ be a finite set of agents and let 0 be the *source*. Networks, where the source supplies the agents in N with some homogeneous good, are considered. Formally a network g over $N^0 = N \cup \{0\}$ is a set of unordered pairs ij where $i, j \in N^0$. Let $N^0(2)$ be the set of unordered pairs, so the cardinality of $N^0(2)$ is $(n+1)n/2$. Moreover $N^0(2)$ is the *complete network* on N^0. Finally let $G^0 = \{g | g \subset N^0(2)\}$ be the set of all networks over N^0.

Two agents i and j are *connected* in g if there is a path $i_1i_2, i_2i_3, \ldots, i_{h-1}i_h$ such that $i_xi_{x+1} \in g$ for $1 \leq x \leq h-1$ where $i = i_1$ and $j = i_h$. A network g is *connected* if i and j are connected in g for all $i, j \in N^0$. A path is a *cycle* if it starts and ends with the same agent. A network is a *tree* if it contains no cycles. A *spanning tree* is a tree connecting all agents in N^0, so there are $(n+1)^{n-1}$ such spanning trees.

For every pair $ij \in N^0(2)$, there is a net-cost $k_{ij} \in \mathbb{R}$ attached to the link between agents i and j. For instance, we may think of such net-costs as the cost (if $k_{ij} > 0$) or benefit (if $k_{ij} < 0$) of establishing the link. For N^0, let the matrix $K \in \mathbb{R}^{N^0(2)}$ be the associated net-cost matrix. Note that this is a generalization of the standard minimum cost spanning tree model where all links are costly (i.e. $k_{ij} \geq 0$ for all $ij \in N^0$).

For a spanning tree p let $v(N,K,p) = \sum_{ij \in p} k_{ij}$ denote the total net-cost of p. A *minimum cost spanning tree* (MCST) is a spanning tree p where the total net-cost $v(N,K,p)$ is minimized over all spanning trees of N^0. For every *cost allocation problem* (N,K) there exists a MCST because the set of spanning trees is finite. The minimal net-cost is denoted $v(N,K)$. There is a unique MCST if all costs k_{ij} are different, but in general there may be several MCSTs: Indeed if all net-costs k_{ij} are equal there are $(n+1)^{n-1}$ MCSTs. In general, let $T(N,K)$ be the set of MCSTs for the problem (N,K).

MCSTs can be found using e.g. the algorithm by Prim (1957): Connect an agent to the source among the set of agents having the lowest source cost. Among the remaining agents, find the set of agents with the lowest cost to the source or the agent already connected to the source. Connect an agent from this set via the relevant link. Continue doing this for the set of remaining agents. In this way the MCST is found in n steps.

Following Bird (1976) there is a unique cooperative game associated with every cost allocation problem (N,K) where the value of coalition $S \subset N$ is the minimum net-cost of connecting all members of S to the source. Therefore all solutions to cooperatives games (such as e.g. the Shapley value, the Nucleolus etc) can be used to allocate costs in cost allocation problems.

Irreducible cost matrices

Consider the reduction of the cost matrix K analyzed in Bird (1976) and Aarts and Driessen (1993). For two matrices K and K', the matrix K is smaller than K' if and only if $k_{ij} \leq k'_{ij}$ for all $i,j \in N^0$. The *irreducible* matrix $C(K)$ associated with cost matrix K is the smallest matrix K' such that $v(N,K') = v(N,K)$ and $k'_{ij} \leq k_{ij}$ for all $i,j \in N^0$. For a cost matrix K and a spanning tree p the irreducible matrix $C(K,p)$ is defined as follows: For every $i,j \in N^0$, let p_{ij} be the unique chain in p from i to j, then $c_{ij} = \max_{xy \in p_{ij}}\{k_{xy}\}$. It is known that if p^* is a MCST, then $C(K) = C(K,p^*)$. Hence if both p^* and q^* are MCSTs then $C(K,p^*) = C(K,q^*) = C(K)$.

It turns out that the irreducible matrix is minimal for MCSTs.

Lemma 1 *For a cost allocation problem (N,K) suppose that p^* is a MCST and that p is an arbitrary spanning tree. Then $C(K,p^*)$ is smaller than*

$C(K,p)$.

Proof: For two spanning trees p^* and p, suppose that there exists $i,j \in N^0$ such that $c_{ij}(K,p^*) > c_{ij}(K,p)$. Then $\max_{xy \in p^*_{ij}} k_{xy} > \max_{x'y' \in p_{ij}} k_{x'y'}$. Therefore there exists a link xy in p^*_{ij} such that $k_{xy} > \max_{x'y' \in p_{ij}} k_{x'y'}$.

Let q be the spanning tree p^* without the link xy. Then in q there is a chain from either i or j to the source. Let A_i, resp. A_j, be the set agents that are linked to i, resp. j, in q. Then there exists a link $x'y'$ in p_{ij} that connects A_i and A_j because $i \in A_i$ and $j \in A_j$. Let q^* be q and the link $x'y'$. Then q^* is a spanning tree and $C(K,q^*)$ is smaller than $C(K,p^*)$ as $c_{ij}(K,q^*) < c_{ij}(K,p^*)$. Hence p^* is not a MCST. Thus if p^* is a MCST and p is a spanning tree, then $C(K,p^*)$ is smaller than $C(K,p)$.

Q.E.D

The irreducible matrix $C(K,p)$ will play a crucial role in the implementation games to be defined in Sections 3 and 5.

Allocation Rules

Let Γ be the set of cost allocation problems and their spanning trees, so $(N,K,p) \in \Gamma$ if and only if (N,K) is a cost allocation problem and p is a spanning tree for (N,K). A cost allocation rule $\phi : \Gamma \to \mathbb{R}^N$ assigns to any cost allocation problem (N,K) and spanning tree p an n-dimensional vector of costs $(\phi_i(N,K,p))_{i \in N}$.

In the sequel we only consider cost allocation rules that are *budget balanced*, *verifiable* and *continuous* as defined below:

- *Budget-balance:* $\sum_{i \in N} \phi_i(N,K,p) = v(N,K)$ for all $p \in T(N,K)$, so cost shares add up to the total cost of the MCST.
- *Verifiability:* $\phi_i(N,K,p) = \phi_i(N,C(K,p),p)$ for all spanning trees p, so cost shares depend on the realized costs of the chosen spanning tree.
- *Continuity:* $\phi(N,K,p)$ is continuous in K.

Budget-balance and continuity are standard properties of cost allocation rules. Verifiability is crucial in the implementation game introduced in the following section. Clearly, budget-balance and verifiability imply that the total cost of any spanning tree is fully allocated by the cost allocation rule.

Examples of well-known cost allocation rules, all satisfying budget-balance, verifiability and continuity, are:

- *The Equal Split Rule:* For all $i \in N$

$$\phi_i^E(N, K, p) = \frac{v(N, C(K, p))}{n}.$$

- *The Bird Rule:* For all $i \in N$

$$\phi_i^B(N, K, p) = k_{ji} = c_{ji}(K, p)$$

  where j is the upstream agent of i in the unique chain from 0 til i in p.

- *The Proportional Rule:* For all $i \in N$

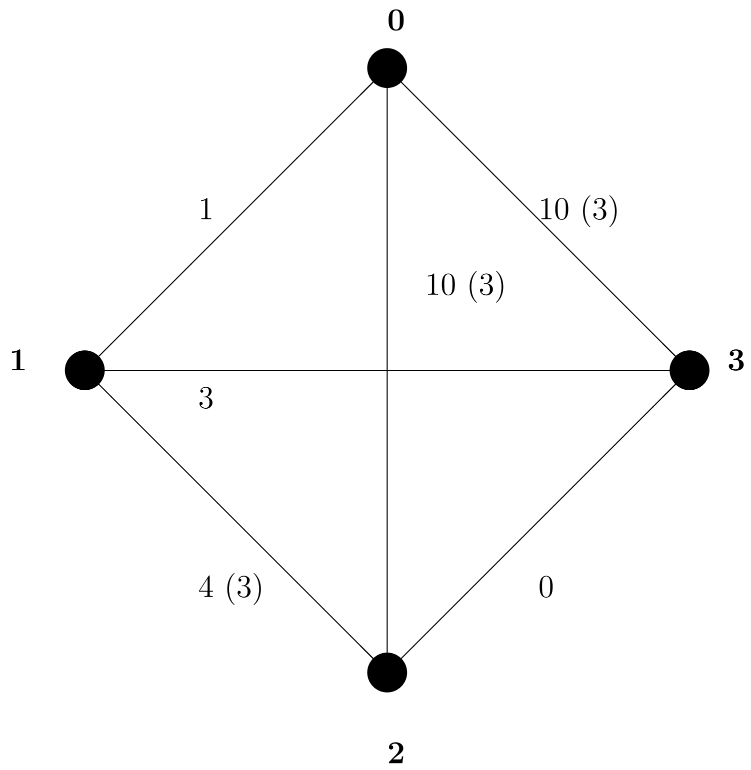
$$\phi_i^P(N, K, p) = \frac{c_{0i}(K, p)}{\sum_{j \in N} c_{0j}(K, p)} v(N, C(k, p)).$$

- *The Shapley Rule:* For all $i \in N$

$$\phi_i^S(N, K, p) = \sum_{S \subset N : i \in S} \frac{(s-1)!(n-s)!}{n!} (v(S, C(K, p)) - v(S \backslash i, C(K, p)))$$

  where $v(S, C(K, p))$ is the minimal cost of connecting all members of S to the source 0 given the projection of $C(K, p)$ on S. Note that the Shapley value in our context coincides with the so-called Folk-solution considered in Bergantinos & Vidal-Puga (2007, 2009) and Bogomolnaia & Moulin (2008).

Example: To demonstrate that these rules can generate quite different cost allocations consider the following case where $N^0 = \{0, 1, 2, 3\}$ and $K = (k_{01}, k_{02}, k_{03}, k_{12}, k_{13}, k_{23}) = (1, 10, 10, 4, 3, 0)$ as illustrated below (irreducible costs are given in brackets).



Clearly, $p = (01, 13, 32)$ is the unique MCST of (N, K) so $C(K, p) = (1, 3, 3, 3, 3, 0)$. Consequently, we get that
$\phi^E(N, C(K, p), p) = (4/3, 4/3, 4/3)$,
$\phi^B(N, C(K, p), p) = (1, 0, 3)$,
$\phi^P(N, C(K, p), p) = (4/7, 12/7, 12/7)$
$\phi^S(N, C(K, p), p) = (1, 3/2, 3/2)$. *End of example*

3 The Game

In this section we introduce a two-stage game between a central planner and a set of agents. The planner wants to connect the agents to the source in the least costly way, but is unaware of the costs of connecting the agents (including the costs of connecting each agent to the source). These link costs, on the other hand, are known by the agents.

We now imagine a scenario where every agent announces all the link costs to the planner who then, based on these announcements, estimates a cost matrix and determines the associated efficient spanning tree (the MCST given the estimation). Once the efficient spanning tree has been constructed and the actual *true* link costs are realized, the planner uses a cost allocation rule to distribute the total costs of the realized network.

Since only costs associated with links in the chosen spanning tree are realized, and therefore truthfully revealed, this raises the question of how to go from the realized costs related to the chosen spanning tree to the entire cost matrix involving all link costs. Here, we assume that the planner, once the true costs along the chosen spanning tree are found, uses the associated irreducible link cost matrix to 'overrule' the remaining cost announcements. It hence becomes natural to restrict attention to cost allocation rules satisfying verifiability.

To be more precise; the implementation game has two stages: in the first stage, the planner announces the rules of the game, which consists of a cost allocation rule and an estimation rule; in the second stage, every agent announces the link costs. Then the planner: i. estimates a cost matrix by use of the estimation rule based on the announcements of the agents; ii. chooses a cost minimizing spanning tree given the estimation, and; iii. allocates realized costs of the chosen spanning tree based on the cost allocation rule.

The rule of the game consists of a cost allocation rule $\phi : \Gamma \to \mathbb{R}^n$ and an estimation rule $\tau : \mathbb{R}^n \to \mathbb{R}$ where τ is used to estimate the cost of every link based on the n agents' announced costs of the link. The estimation rule τ is supposed to have values between the minimum and maximum announcement, i.e. $\tau(\sigma_{ij}^1, \dots, \sigma_{ij}^n) \in [\min\{\sigma_{ij}^1, \dots, \sigma_{ij}^n\}, \max\{\sigma_{ij}^1, \dots, \sigma_{ij}^n\}]$, and be downward and upward unbounded, i.e. $\lim_{\sigma_{ij}^z \to -\infty} \tau(\sigma_{ij}^1, \dots, \sigma_{ij}^n) = -\infty$ and $\lim_{\sigma_{ij}^z \to \infty} \tau(\sigma_{ij}^1, \dots, \sigma_{ij}^n) = \infty$ for every $z \in N$.

For a list of individual announcements of link costs $\sigma \in (\mathbb{R}^{N^0(2)})^n$, where $\sigma = (\sigma^1, \dots, \sigma^n)$ and $\sigma^i = (\sigma_{01}^i, \dots, \sigma_{n-1n}^i)$, the estimated cost of the link ij is $\tau(\sigma_{ij}^1, \dots, \sigma_{ij}^n)$. The associated cost matrix is denoted $K^e(\sigma)$. For every $p \in T(N, K^e(\sigma))$, the cost shares are given by $(\phi_i(N, C(K, p), p))_{i \in N}$. Every $p \in T(N, K^e(\sigma))$ is equally likely, so the expected cost of agent i is

$$\sum_{p \in T(N, K^e(\sigma))} \frac{1}{|T(N, K^e(\sigma))|} \phi_i(N, C(K, p), p).$$

For fixed rules of the game, agents aim at minimizing their expected costs.

Obviously, if the agents know how the planner will allocate the realized link costs they may act strategically when they announce the link costs. By lying, an agent can make the planner choose a certain spanning tree that makes him better off.

Example: Consider a case where the planner uses the Bird rule and the average of the announcements as an estimation rule: Let $N = \{1, 2\}$ and the true link costs be given by $k_{01} = 2$, $k_{02} = 2$ and $k_{12} = 1$. Clearly, if both agents announce the true link costs (i.e. $\sigma^2 = \sigma^1 = (k_{01}, k_{02}, k_{12}) = (2, 2, 1)$ the planner selects one of the two (true) MCSTs $p = \{01, 12\}$ or $q = \{02, 21\}$ with expected cost of $3/2$ for both agents using the Bird rule. Now, since agent 1 knows that the planner will use the Bird rule and the average as the estimation rule he will obviously try to manipulate the planner. Assume that agent 2 announces the truth, then agent 1 will announce e.g. $\sigma^1 = (\sigma_{01}, k_{02}, k_{12}) = (\sigma_{01}, 2, 1)$, where $\sigma_{01} > 2$, in order to make the planner choose the spanning tree $p^* = \{02, 21\}$ with Bird allocation $(1, 2)$ which clearly makes agent 1 better off. *End of example*

In the example above manipulation is caused by the fact that the Bird rule depends on the specific form of the spanning tree (and not only the irreducible cost matrix which is identical for all efficient spanning trees).

We say that a cost allocation rule is *Spanning Tree Independent* provided that for two spanning trees p and q, if $C(K, q) = C(K, p)$, then $\phi_i(N, K, p) = \phi_i(N, K, q)$ for all i.

Clearly the Equal Split, the Proportional and the Shapley rule are all Spanning Tree Independent. Yet, Spanning Tree Independence is not sufficient to ensure that all agents announce the truth as demonstrated by the following example.

Example: Consider the Proportional Rule ϕ^P and let $N^0 = \{0, 1, 2, 3\}$ and

$$K = (k_{01}, k_{02}, k_{03}, k_{12}, k_{13}, k_{23}) = (1, 10, 10, 2, 3, 0).$$

Clearly, $C(K) = (1, 2, 2, 2, 2, 0)$ with MCST $p = (01, 12, 23)$ yielding

$$\phi_1^P(N, C(K, p), p) = 3/5.$$

Now suppose that agents 2 and 3 announce the true link costs and let the estimation rule be e.g. the average of the announcements, then agent 1 can gain by announcing $\sigma_1 = (\sigma_{01}, \sigma_{12}, \sigma_{13}) = (1,4,3)$ since the planner will now choose the efficient spanning tree $q = (01, 13, 32)$ making $C(K^e, q) = (1,3,3,3,3,0)$ as in the first example where $\phi_1^P(N, C(K^e, q), q) = 4/7 < 3/5$. *End of example*

Consequently, it appears that some form of monotonicity in cost shares is crucial for truthful reporting of link costs.

- **Monotonicity:** A cost allocation rule is *monotonic* provided that for two spanning trees p and q, if $C(K,q) \geq C(K,p)$ then

$$\phi_i(N, C(K,q), q) \geq \phi_i(N, C(K,p), p)$$

  for all $i \in N$.

Note that Monotonicity implies Spanning Tree Independence. Indeed, for two spanning trees p and q, if $C(K,q) = C(K,p)$, then $C(K,q) \geq C(K,p)$, so $\phi_i(N, C(K,q), q) \geq \phi_i(N, C(K,p), p)$, and $C(K,p) \geq C(K,q)$, so $\phi_i(N, C(K,p), p) \geq \phi_i(N, C(K,q), q)$. Therefore

$$\phi_i(N, C(K,q), q) = \phi_i(N, C(K,p), p)$$

.

As demonstrated by the examples above neither the Bird rule nor the Proportional rule satisfy Monotonicity. However, it is clear that the class of Monotonic rules is non-empty as recorded in the Lemma below.

Lemma 2 *The Equal Split and the Shapley Rules satisfy Monotonicity.*

Proof: The equal split rule is Monotonic by definition. From Bogomolnaia & Moulin (2008) and Hougaard, Moulin & Østerdal (2008) it is known that the Shapley Rule can be determined as

$$\phi_i^S(N, K, p) \; = \phi_i^S(N, C(K,p), p) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \min_{j \in \mathcal{P}(i,\pi)} c_{ij}(K,p)$$

where Π^N is the set of orderings of N and $\mathcal{P}(i,\pi) = \{0\} \cup \{j \in N | \pi(j) < \pi(i)\}$. Hence, the Shapley value is Monotonic. *Q.E.D*

Indeed, the class of monotonic rules is rather large, see e.g. Bergantinos and Kar (2007).

4 Implementation of MCSTs

As the first main result we have the following theorem.

Theorem 1 *Truth-telling is a Nash equilibrium if and only if the allocation rule is Monotonic.*

Proof: Suppose that an allocation rule is Monotonic. Then no agent can gain by deviating from truth-telling, because the irreducible matrix is minimized for truth-telling according to Lemma 1.

Suppose that an allocation rule is not Monotonic. Then there exists an allocation problem (N,K) where $C(K,q) \geq C(K,p)$ and $\phi_z(N, C(K,q), q) < \phi_z(N, C(K,p), p)$ for some agent $z \in N$. Let a new problem (N,K') be defined by $k'_{ij} = k_{ij}$ for $ij \in p$, $k'_{ij} = k_{ij} + \varepsilon$ for $ij \in q \setminus p$ where $\varepsilon > 0$ and $k'_{ij} = \max_{i'j'}\{k_{i'j'}\} + 1$ otherwise. Note that p is the unique MCST in (N,K') and that $\phi_z(N,K',q) < \phi_z(N,K',p)$ for sufficiently small ε by continuity. Suppose that all agents except z are announcing the truth. Then z can gain by announcing the truth for all links in q and sufficiently high costs for all other links. Indeed, since τ is upward unbounded q will be chosen by the planner because q is the unique MCST for $(N, K^e(\sigma))$.

Q.E.D

Remark: The theorem remains valid if the notion of equilibrium is strengthened from Nash equilibrium to strong Nash equilibrium. Moreover, the proof demonstrates that an attempt by the planner to enforce truth-telling by punishing lies will fail. Indeed, the manipulation of the spanning tree from p to q by agent z will not be revealed because he only lies about non-realized link costs.

End of remark

Now, we can further show that if the planner uses a Monotonic rule then all equilibria implements the efficient network.

Theorem 2 *If an allocation rule is Monotonic and σ is a Nash equilibrium, then $p \in T(N, K^e(\sigma))$ implies that $p \in T(N, K)$.*

Proof: Suppose that an allocation rule is Monotonic, σ is a Nash equilibrium and $p \in T(N, K^e(\sigma))$ but $p \notin T(N, K)$. Then

$$\sum_{i \in N} \sum_{p \in T(N, K^e(\sigma))} \frac{1}{|T(N, K^e(\sigma))|} \phi_i(N, K, p) > \sum_{i \in N} \sum_{q \in T(N, K)} \frac{1}{|T(N, K)|} \phi_i(N, K, q)$$

. Therefore

$$\sum_{p \in T(N, K^e(\sigma))} \frac{1}{|T(N, K^e(\sigma))|} \phi_z(N, K, p) > \sum_{q \in T(N, K)} \frac{1}{|T(N, K)|} \phi_z(N, K, q)$$

for some z. Suppose that all agents except z are announcing σ_{-z}. Let $q \in T(N, K)$, then z can gain by announcing sufficiently low costs for all links in q and sufficiently high costs for all other links. Indeed, since τ is downward and upward unbounded q will be chosen by the planner because q is the unique MCST.

Q.E.D

In other words, according to Theorems 1 and 2, by choosing a Monotonic allocation rule the planner can ensure that only MCSTs are implemented in equilibrium.

Remark: The proof demonstrates that lies are important in eliminating equilibria resulting in implementation of spanning trees that are not MCSTs. Therefore an attempt of the planner to enforce truth-telling by punishing lies can result in implementation of spanning trees that are not MCSTs.

End of remark

Furthermore, since Monotonicity implies Spanning Tree Independence we record that in any strategyproof Nash equilibrium the allocation rule must be Spanning Tree Independent.

Corollary 1 *If truth-telling constitutes a Nash equilibrium, then the cost allocation rule is Spanning Tree Independent.*

In practice agents need not know all the link costs. Thus we next consider an implementation scenario where agents are restricted to announce their own link costs only. We show that Theorem 1 extends to this scenario under an additional assumption. Thereby we argue that if agents only know their own link costs and the rule is monotonic, then truth-telling is a focal point in the sense that if an agent believes that all other agents are telling the truth, then truth-telling is an optimal strategy for the agent.

5 The Restricted Game

In this section we modify the two-stage game of Section 3, between a central planner and a set of agents. We now imagine a scenario where every agent only announces their own link costs to the planner who then, based on these announcements, estimates a cost matrix and determines the associated efficient network (the MCST). Again, once the efficient spanning tree has been constructed and the (true) link costs are realized, the planner uses a cost allocation rule to distribute the total costs of the realized network.

To be more precise; The rules of the game consists of a cost allocation rule $\phi : \Gamma \to \mathbb{R}^n$ and an estimation rule $\tau : \mathbb{R}^2 \to \mathbb{R}$ where τ is used to estimate the cost of every link ij based on the announcements of both agents i and j. The estimation rule τ is supposed to take values between the minimum and maximum announcement, i.e. $\tau(\sigma^i_{ij}, \sigma^j_{ij}) \in [\min\{\sigma^i_{ij}, \sigma^j_{ij}\}, \max\{\sigma^i_{ij}, \sigma^j_{ij}\}]$, and be downward as well as upward unbounded, i.e. $\lim_{\sigma^z_{ij} \to \infty} \tau(\sigma^i_{ij}, \sigma^i_{ij}) = \infty$ and $\lim_{\sigma^z_{ij} \to -\infty} \tau(\sigma^i_{ij}, \sigma^i_{ij}) = -\infty$ for both $z = i$ and $z = j$. Since the planner does not announce the costs of any links, there is only one announcement for every source cost and the estimation is simply identical to this announcement.

Unfortunately the result of Theorem 1 does not seem to generalize to the restricted game in a straightforward way. However, Monotonicity still implies strategyproofness as recorded by the corollary below.

Corollary 2 *If the allocation rule is Monotonic then Truth-telling is a Pareto optimal Nash equilibrium.*

Proof: It follows from Lemma 1 that monotonicity implies strategyproof Nash equilibrium. In strategyproof Nash equilibrium MCSTs are selected which implies Pareto optimality.

Q.E.D

It is tempting to conjecture that the reverse of Corollary 2 should be true as well. However, imagine that some agents cost share is non-monotone in link costs that he cannot influence (e.g. agent i's cost share can be non-monotone in k_{jl} where $j, l \neq i$). Then truth-telling may still constitute a Nash equilibrium because the agent cannot gain by manipulating his own announced link costs.

Hence, we will restrict attention to allocation rules which ensure that if some agent i gets a lower cost share in a spanning tree that is not a MCST then there exists another spanning tree differing from the MCST with only one link and some neighbor agent who also gets a lower cost.

Formally, for a spanning tree p let $\delta(i, j, p)$ be the unique neighbor of i in the chain from i to j in p_{ij}. For two graphs p and q let $p \setminus q$ be the links in p but not in q and let $|p|$ be the number of links in p.

- *Responsiveness:* For two spanning trees p and q, where $p \in T(N, K)$, suppose that $\phi_i(N, C(K, q), q) < \phi_i(N, C(K, p), p)$ for some i. Then there exist two spanning trees p' and q', where $p' \in T(N, K)$ and $|p' \setminus q'| = |q' \setminus p'| = 1$, such that $\phi_{i'}(N, C(K, q'), q') < \phi_{i'}(N, C(K, p'), p')$ for some i' with $\cup_{j \in N^0} \delta(i', j, q') \setminus \cup_{j \in N^0} \delta(i', j, p') \neq \emptyset$ or $\cup_{j \in N^0} \delta(i', j, p') \setminus \cup_{j \in N^0} \delta(i', j, q') \neq \emptyset$.

Note that Monotonic allocation rules are Responsive because if $C(K, q) \geq C(K, p)$, then $\phi_i(N, C(K, q), q) \geq \phi_i(N, C(K, p), p)$ for all i by Monotonicity. But, also rules that violate Monotonicity may still satisfy Responsiveness as for example the Bird rule.

Theorem 3 *The Bird rule ϕ^B satisfies Responsiveness*

Proof: Let j be the agent for which k_{ij} is minimal for i and assume that $\phi_i^B(N, C(K,q), q) < \phi_i^B(N, C(K,p), p)$ where $p \in T(N,K)$ and q is a spanning tree. Consequently, j cannot be the predecessor of i in p. Let j' be the predecessor of i in p. Note that j must also be located downstream of i in p since otherwise there would exist a spanning tree r identical to p except replacing the link ij' with ij such that $v(N,K,r) < v(N,K,p)$ contradicting that p is MCST. Now, let $p' = p$ and let q' be a spanning tree identical to p except for changing the link $j'i$ to the link $j'j$ (i.e. $|p' \setminus q'| = |q' \setminus p'| = 1$). We conclude that ϕ^B is responsive. *Q.E.D*

We now establish a characterization result for the restricted game.

Theorem 4 *Suppose that the cost allocation rule is Responsive. Then truth-telling is a Nash equilibrium if and only if the cost allocation rule is Monotonic.*

Proof: It follows from Corollary 2 that if a Responsive allocation rule is Monotonic, then truth-telling is a Nash equilibrium.

Conversely, suppose that the cost allocation rule is not Monotonic, so there exist a pair of spanning trees p and q, where $C(K,q) \geq C(K,p)$, such that $\phi_i(N, C(K,q), q) < \phi_i(N, C(K,p), p)$ for some i. Suppose that $v(N,K,p) \leq v(N,K,q)$ and let K' be defined by $k'_{ij} = k_{ij}$ for $ij \in p$ or $ij \in q$ and $k'_{ij} > \max_{i'j' \in N^0} k_{i'j'}$ otherwise. Then $p \in T(N,K')$. Since ϕ is Responsive there exist spanning trees p' and q', where $p' \in T(N,K')$, where $|p' \setminus q'| = |q' \setminus p'| = 1$, such that $\phi_{i'}(N, C(K',q'), q') < \phi_{i'}(N, C(K',p'), p')$ for some i' with $\cup_{j \in N^0} \delta(i', j, q') \setminus \cup_{j \in N^0} \delta(i', j, p') \neq \emptyset$ or $\cup_{j \in N^0} \delta(i', j, p') \setminus \cup_{j \in N^0} \delta(i', j, q') \neq \emptyset$.

Let a new cost allocation problem (N, K'') be defined by $k''_{ij} = k'_{ij}$ for $ij \in p'$ and $k''_{ij} = k'_{ij} + \varepsilon$ for $ij \in q' \setminus p'$ and $k''_{ij} > \max_{i',j' \in N^0} k'_{i'j'}$ otherwise. Then $T(N,K'') = \{p'\}$ and $\max_{ij \in p'} k''_{ij} < \min_{ij \in q' \setminus p'} k''_{ij}$. Moreover there exists $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon < \bar{\varepsilon}$, then $\phi_{i'}(N, C(K'',q'), q') < \phi_{i'}(N, C(K'',p'), p')$ by continuity of ϕ and q' is cheapest spanning tree that is not a MCST. Suppose that all agents except agent i' use truth-telling as strategy. Then, since τ is unbounded, there exists a strategy for agent i', where link costs in p' but not in q' are announced more expensive than actual costs and link costs in q' but not in p' are announced less expensive than actual costs, results in q'. Therefore truth-telling is not a Nash equilibrium.

Q.E.D

6 Final Remarks

We have shown that in situations where agents are able to react strategically the planner ought to make sure that the allocation rule is Monotonic in order to implement a MCST. Several rules are monotonic, in particular, the Shapley rule (which coincides with the Folk-solution in our model).

Characterization results using weaker versions of Monotonicity as well as an extended analysis of the class of Monotonic and/or the class of Monotonic and Responsive allocation rules remain topics for further research.

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